

On excluded minors of connectivity 2 for the class of frame matroids

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Abstract

We investigate the set of excluded minors of connectivity 2 for the class of frame matroids. We exhibit a list \mathcal{E} of 18 such matroids, and show that if N is such an excluded minor, then either $N \in \mathcal{E}$ or N is a 2-sum of $U_{2,4}$ and a 3-connected non-binary frame matroid.

Keywords: biased graphs, frame matroids, excluded minors.

MSC: 05C22, 05B35.

1 Introduction

A matroid is *frame* if it may be extended so that it contains a basis B (its *frame*) such that every element is spanned by two elements of B . Frame matroids are a natural generalisation of graphic matroids. Indeed, the cycle matroid $M(G)$ of a graph $G=(V, E)$ is naturally extended by adding V as a basis, and declaring each non-loop edge to be minimally spanned by its endpoints. Zaslavski [13] has shown that the class of frame matroids is precisely that of matroids arising from *biased graphs* (whence these have also been called *bias* matroids), as follows. A *biased graph* Ω consists of a pair (G, \mathcal{B}) , where G is a graph and \mathcal{B} is a collection of cycles of G , called *balanced*, such that no theta subgraph contains exactly two balanced cycles; a *theta* graph consists of a pair of distinct vertices and three internally disjoint paths between them. We say such a collection \mathcal{B} satisfies the *theta property*. The membership or non-membership of a cycle in \mathcal{B} is its *bias*; cycles not in \mathcal{B} are *unbalanced*.

Let M be a frame matroid on ground set E , with frame B . By adding elements in parallel if necessary, we may assume $B \cap E = \emptyset$. Hence for some matroid N ,

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$M = N \setminus B$ where B is a basis for N and every element $e \in E$ is spanned by a pair of elements in B . Let G be the graph with vertex set B and edge set E , in which e is a loop with endpoint f if e is in parallel with $f \in B$, and otherwise e is an edge with endpoints $f, f' \in B$ if $e \in \text{cl}\{f, f'\}$. Setting $\mathcal{B} = \{C \mid C \text{ is a cycle for which } E(C) \text{ is a circuit of } M\}$ yields a biased graph (G, \mathcal{B}) , and the circuits of M are precisely those sets of edges inducing one of: (1) a balanced cycle, (2) two edge-disjoint unbalanced cycles intersecting in just one vertex, (3) two vertex-disjoint unbalanced cycles along with a path connecting them, or (4) a theta subgraph in which all three cycles are unbalanced [13]. We call a subgraph as in (2) or (3) a *pair of handcuffs*, *tight* or *loose*, respectively. We say such a biased graph (G, \mathcal{B}) *represents* the frame matroid M , and write $M = F(G, \mathcal{B})$.

Observe that for a biased graph (G, \mathcal{B}) , if \mathcal{B} contains all cycles in G , then $F(G, \mathcal{B})$ is the cycle matroid $M(G)$ of G . We therefore view a graph as a biased graph with all cycles balanced. At the other extreme, when no cycles are balanced $F(G, \emptyset)$ is the bicircular matroid of G , introduced by Simões-Pereira [9] and further investigated by Matthews [6], Wagner [10], and others (for instance, [5, 7]). Frame matroids also include Dowling geometries [4] (see also [14]).

A class of matroids is *minor-closed* if every minor of a matroid in the class is also in the class. For any minor-closed family, there is a set of *excluded minors* consisting of those matroids not in the family all of whose proper minors are in the family. The class of frame matroids is minor-closed. Little is known about excluded minors for the class of frame matroids; Zaslavski has exhibited several in [13]. The class of bicircular matroids is minor-closed; DeVos, Goddyn, Mayhew, and Royle [2] have shown that an excluded minor for the class of bicircular matroids has less than 16 elements, and thus that the set of excluded minors for this class is finite. Perhaps, like graphic and bicircular matroids, the larger class of frame matroids may also be characterised by a finite list of excluded minors. On the other hand, as we have shown elsewhere [3], there are natural minor-closed families of frame matroids whose sets of excluded minors are infinite. Perhaps, like the class of matroids representable over the reals, the set of excluded minors for frame matroids is infinite. In this paper, we begin by seeking to determine those excluded minors for the class of frame matroids that are not 3-connected. We come close, determining a set \mathcal{E} of 18 particular excluded minors for the class, and show that any other excluded minor of connectivity 2 for the class has a special form. We prove:

Theorem 1.1. *Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. Then either M is isomorphic to a matroid in \mathcal{E} or M is the 2-sum of a 3-connected non-binary frame matroid and $U_{2,4}$.*

The remainder of this paper is organised as follows. We first discuss some of the key concepts we need for our investigation. In Section 2 we discuss 2-sums of frame matroids and of biased graphs, and provide a characterisation of when a 2-sum of two frame matroids is frame. This is enough for us to determine the first nine excluded

minors on our list, and to drastically narrow our search for more. These tasks are accomplished in Section 3. In particular, we investigate some key properties any excluded minor not yet on our list must have. In Section 4 we complete the proof of Theorem 1.1, determining the remaining excluded minors in our list.

Theorem 1.1 give a strong structural description of excluded minors that are not 3-connected. However, the investigation remains incomplete — the final case remaining is to determine those excluded minors of the form captured in the second part of the statement of Theorem 1.1. We anticipate that the analysis required to complete this final case will be at least as long and technical as that required here, and that the result will be at least a doubling of the number of excluded minors on our list, but that the list will remain finite.

We close this preliminary section by noting that in the course of proving Theorem 1.1, we discover an operation analogous to a Whitney twist in a graph, which we call a *twisted flip*. Just as a Whitney twist of a graph G produces a (generally) non-isomorphic graph whose cycle matroid is isomorphic to the cycle matroid of G , a twisted flip of a biased graph (G, \mathcal{B}) produces a (generally) non-isomorphic biased graph (G', \mathcal{B}') with $F(G', \mathcal{B}') \cong F(G, \mathcal{B})$. This operation is described toward the end of Section 1.3.

1.1 Standard notions: biased graphs, minors, connectivity

For a frame matroid M represented by a biased graph $\Omega = (G, \mathcal{B})$, we denote throughout by $E = E(M) = E(G)$ the common ground set of M and edge set of G . When it is important to distinguish an edge which is not a loop from one that is, we refer to an edge having distinct endpoints as a *link*. There are minor operations we may perform on (G, \mathcal{B}) that correspond to minor operations in M , as follows [12]. For an element $e \in E$, *delete* e from (G, \mathcal{B}) by deleting e from G and removing from \mathcal{B} every cycle containing e . To *contract* e , there are three cases: If e is a balanced loop, $(G, \mathcal{B})/e = (G, \mathcal{B}) \setminus e$. If e is a link, contract e in G and declare a cycle C to be balanced if either $C \in \mathcal{B}$ or $E(C) \cup \{e\}$ forms a cycle in \mathcal{B} . If e is an unbalanced loop with endpoint u , then $(G, \mathcal{B})/e$ is the biased graph obtained from (G, \mathcal{B}) as follows: e is deleted, all other loops incident to u become balanced, and links incident to u become unbalanced loops incident to their other endpoint. A *minor* of (G, \mathcal{B}) is any biased graph obtained by a sequence of deletions and contractions. It is readily checked that these minor operations on biased graphs preserve the theta property, and that they agree with matroid minor operations on their frame matroids; that is, for any element $e \in E$, $F(G, \mathcal{B}) \setminus e = F((G, \mathcal{B}) \setminus e)$ and $F(G, \mathcal{B})/e = F((G, \mathcal{B})/e)$ (this shows that the class of frame matroids is minor closed).

For a biased graph $\Omega = (G, \mathcal{B})$ we say G is the *underlying graph* of Ω . We write $\Omega[X]$ or $G[X]$ to denote the biased subgraph of (G, \mathcal{B}) induced by the edges in X that has balanced cycles just those cycles in \mathcal{B} whose edge set is contained in X . If $G[X]$ contains no unbalanced cycle, it is *balanced*; otherwise it is *unbalanced*. If $G[X]$ contains no balanced cycle, it is *contrabalanced*. We denote by $V(X)$ the

set of vertices incident with an edge in X , and by $b(X)$ the number of balanced components of $G[X]$. It follows from the definitions that for a frame matroid M represented by biased graph (G, \mathcal{B}) , the rank of X in M is $r(X) = |V(X)| - b(X)$.

A *separation* of a graph $G=(V, E)$ is a pair of edge disjoint subgraphs G_1, G_2 of G with $G = G_1 \cup G_2$. The *order* of a separation is $|V(G_1) \cap V(G_2)|$. A separation of order k is a *k-separation*. If both $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$ are non-empty, then the separation is *proper*. If G has no proper separation of order less than k , then G is *k-connected*. The least integer k for which G has a proper *k*-separation is the *connectivity* of G . A partition (X, Y) of E naturally induces a separation $G[X], G[Y]$ of G , which we also denote (X, Y) . We call X and Y the *sides* of the separation. The *connectivity function* of G is the function λ_G that to each partition (X, Y) of E assigns the order of its corresponding separation; that is, $\lambda_G(X, Y) = |V(X) \cap V(Y)|$.

A *k-separation* of a biased graph $\Omega=(G, \mathcal{B})$ is a *k*-separation of its underlying graph G , and the *connectivity* of Ω is that of G . The *connectivity function* λ_Ω of Ω is that of G .

A *separation* of a matroid M is a partition of its ground set E into two subsets X, Y ; it is also denoted (X, Y) , with X and Y the *sides* of the separation. The *order* of a separation (X, Y) of a matroid is $r(X) + r(Y) - r(E) + 1$. A separation of order k with both $|X|, |Y| \geq k$ is a *k-separation*. If M has no *l*-separation with $l < k$, then M is *k-connected*. The *connectivity* of M is the least integer k such that M has a *k*-separation, provided one exists (otherwise we say the connectivity of M is infinite). Evidently, M is connected if and only if M has no 1-separation. The *connectivity function* of a matroid M on ground set E is the function λ_M that assigns to each separation (X, Y) of E its order; that is, $\lambda_M(X, Y) = r(X) + r(Y) - r(M) + 1$.

Let M be a frame matroid represented by a biased graph Ω . The following facts regarding the relationship between the order of a separation (X, Y) in M and the order of (X, Y) in Ω will be used extensively throughout. In general, a separation has different orders in Ω and $F(\Omega)$. However, if the sides of a separation are connected in the graph then this difference is at most one. To see this, let (X, Y) be a partition of E . The order of (X, Y) in M is

$$\begin{aligned} \lambda_M(X, Y) &= r(X) + r(Y) - r(M) + 1 \\ &= |V(X)| - b(X) + |V(Y)| - b(Y) - (|V| - b(E)) + 1 \\ &= |V(X) \cap V(Y)| - b(X) - b(Y) + b(E) + 1 \\ &= \lambda_\Omega(X, Y) - b(X) - b(Y) + b(E) + 1. \end{aligned} \tag{1}$$

Suppose both $\Omega[X]$ and $\Omega[Y]$ connected. If Ω is balanced, we have $\lambda_M(X, Y) = \lambda_\Omega(X, Y)$. If Ω is unbalanced, we have

1. if both $\Omega[X]$ and $\Omega[Y]$ are unbalanced, $\lambda_M(X, Y) = \lambda_\Omega(X, Y) + 1$,
2. if one of $\Omega[X]$ or $\Omega[Y]$ is balanced while the other is unbalanced, then $\lambda_M(X, Y) = \lambda_\Omega(X, Y)$, and

3. if both $\Omega[X]$ and $\Omega[Y]$ are balanced, then $\lambda_M(X, Y) = \lambda_\Omega(X, Y) - 1$.

Moreover, (1) immediately implies that if M is connected, then Ω must be connected: if there is a partition (X, Y) of E with $\lambda_\Omega(X, Y) = 0$, then $\lambda_M(X, Y) = 1$. The converse need not hold: a frame matroid represented by a connected biased graph may be disconnected. Indeed, let $M = F(\Omega)$, where Ω is connected, and suppose (X, Y) is a 1-separation of M , with both $\Omega[X]$ and $\Omega[Y]$ connected. If Ω is balanced, then $\lambda_M(X, Y) = \lambda_\Omega(X, Y) = 1$. Otherwise, by (1) one of the following holds:

- $\lambda_\Omega(X, Y) = 1$, and precisely one of $\Omega[X]$ or $\Omega[Y]$ is balanced;
- $\lambda_\Omega(X, Y) = 2$, and each of $\Omega[X]$ and $\Omega[Y]$ are balanced.

Throughout, matroids and biased graphs are finite; graphs may have loops and parallel edges. We often make no distinction between a subset of elements A of a matroid $M = F(G, \mathcal{B})$, the subset of edges of G representing an edge in A , and the biased subgraph $G[A]$ induced by A .

1.2 Excluded minors are connected, simple, and cosimple

Having established the standard vocabulary of biased graphs and connectivity, we may immediately make the observations that an excluded minor is connected, simple, and cosimple.

Observation 1.2. *If M is an excluded minor for the class of frame matroids, then M is connected.*

We denote the direct sum of two matroids M and N by $M \oplus N$. Evidently, if Ω and Ψ are biased graphs, then the disjoint union $\Omega \dot{\cup} \Psi$ of Ω and Ψ represents $F(\Omega) \oplus F(\Psi)$. We denote the restriction of a matroid M to a subset $A \subseteq E(M)$ by $M|A$. If $M = F(\Omega)$, then clearly $\Omega[A]$ is a biased graph representing $M|A$.

Proof of Observation 1.2. Suppose to the contrary that M is an excluded minor, and that M has a 1-separation (A, B) . Then M is the direct sum of its restrictions to each of A and B . By minimality, each of $M|A$ and $M|B$ are frame. Let Ω and Ψ be biased graphs representing $M|A$ and $M|B$ respectively, and let $\Omega \dot{\cup} \Psi$ denote the biased graph which is the disjoint union of Ω and Ψ . Then $M = M|A \oplus M|B = F(\Omega) \oplus F(\Psi) = F(\Omega \dot{\cup} \Psi)$, so M is frame, a contradiction. \square

Observation 1.3. *Let M be an excluded minor for the class of frame matroids. Then M is simple and cosimple.*

Proof. Suppose M has a loop e . By minimality, there is a biased graph (G, \mathcal{B}) representing $M \setminus e$. Adding a balanced loop labelled e incident to any vertex of G yields a biased graph representing M , a contradiction. Similarly, if M has a

coloop f , consider a biased graph (G, \mathcal{B}) representing M/f . Adding a new vertex w , choosing any vertex $v \in V(G)$, and adding edge $f = vw$ to G yields a biased graph representing M , a contradiction.

Now suppose M has a two-element circuit $\{e, f\}$. Let (G, \mathcal{B}) be a biased graph representing $M \setminus e$. If f is a link in G , say $f = uv$, then let G' be the graph obtained from G by adding e in parallel with f so e also has endpoints u and v , and let $\mathcal{B}' = \mathcal{B} \cup \{C \setminus f \cup e \mid f \subset C \in \mathcal{B}\}$. If f is an unbalanced loop in G , say incident to $u \in V(G)$, then let G' be the graph obtained from G by adding e as an unbalanced loop also incident with u , and let $\mathcal{B}' = \mathcal{B}$. Then $M = F(G', \mathcal{B}')$, a contradiction.

Similarly, if e and f are elements in series in M , let (G, \mathcal{B}) be a biased graph representing M/e . If f is a link in G , say $f = uv$, then let G' be the graph obtained from G by deleting f , adding a new vertex w , and putting $f = uw$ and $e = vw$; let $\mathcal{B}' = \{C \mid C \in \mathcal{B} \text{ or } C/e \in \mathcal{B}\}$. If f is an unbalanced loop in G , say incident to $u \in V(G)$, let G' be the graph obtained from G by deleting f , adding a new vertex w , and adding edges e and f in parallel, both with endpoints u, w ; let $\mathcal{B}' = \mathcal{B}$ (so $\{e, f\}$ is an unbalanced cycle). Again, then $M = F(G', \mathcal{B}')$, a contradiction. \square

1.3 Working with biased graphs

Before determining further properties of excluded minors, we need to develop some tools and establish some basic facts about biased graphs. If X, Y are subgraphs of a graph G , an X - Y *path* in G is a path that meets $X \cup Y$ exactly in its endpoints, with one endpoint in X and the other in Y .

Rerouting. Let G be a graph, let P be a path in G , and let Q be a path internally disjoint from P linking two vertices $x, y \in V(P)$. We say the path P' obtained from P by replacing the subpath of P linking x and y with Q is obtained by *rerouting* P along Q .

Observation 1.4. *Given two u - v paths P, P' in a graph, P may be transformed into P' by a sequence of reroutings.*

Proof. To see this, suppose P and P' agree on an initial segment from u . Let x be the final vertex on this common initial subpath. If $x = v$, then $P = P'$, so assume $x \neq v$. Let y be the next vertex of P' following x that is also in P . Denote the subpath of P' from x to y by Q . Since y is different from x , the path obtained by rerouting P along Q has a strictly longer common initial segment with P' than P . Continuing in this manner, eventually $x = v$, and P has been transformed into P' . \square

The relevance of this for us is the following simple fact. If a subpath R of a path P is rerouted along Q , and the cycle $R \cup Q$ is balanced, we refer to this as *rerouting along a balanced cycle*. If C is a cycle, x, y are distinct vertices in C , P is an x - y path contained in C , Q is an x - y path internally disjoint from C , and the

cycle $P \cup Q$ is balanced, we say the cycle C' obtained from C by rerouting P along Q is obtained from C by *rerouting along a balanced cycle*. The following fact will be used extensively.

Lemma 1.5. *If C' is obtained from C by rerouting along a balanced cycle, then C and C' have the same bias.*

Proof. Since $C \cup Q$ is a theta subgraph, this follows immediately from the theta property. \square

Signed graphs. A *signed* graph consists of a graph G together with a distinguished subset of edges $\Sigma \subseteq E(G)$ called its *signature*. A signed graph naturally gives rise to a biased graph (G, \mathcal{B}_Σ) in which a cycle $C \in \mathcal{B}_\Sigma$ if and only if $|E(C) \cap \Sigma|$ is even (it is immediate that \mathcal{B}_Σ satisfies the theta property). We say that an arbitrary biased graph (G, \mathcal{B}) is a signed graph if there exists a set $\Sigma \subseteq E(G)$ so that $\mathcal{B}_\Sigma = \mathcal{B}$. The following gives a characterisation of when this occurs.

Proposition 1.6. *A biased graph is a signed graph if and only if it contains no contrabalanced theta subgraph.*

Proof. First suppose that (G, \mathcal{B}) is a signed graph, and choose $\Sigma \subseteq E(G)$ so that $\mathcal{B}_\Sigma = \mathcal{B}$. If P_1, P_2, P_3 are three internally disjoint paths forming a theta subgraph in G then two of $|E(P_1) \cap \Sigma|$, $|E(P_2) \cap \Sigma|$, and $|E(P_3) \cap \Sigma|$ have the same parity, and these paths will form a balanced cycle. Thus, every theta subgraph contains a balanced cycle, and thus no contrabalanced theta subgraph exists.

To prove the converse, let (G, \mathcal{B}) be a biased graph which has no contrabalanced theta subgraph. We may assume G is connected; if not, apply the following argument to each component of G . Let T be a spanning tree of G and define Σ by the following rule:

$$\Sigma = \{e \in E(G) \setminus E(T) \mid \text{the unique cycle in } T + e \text{ is unbalanced}\}.$$

We claim that $\mathcal{B}_\Sigma = \mathcal{B}$. To prove this, we will show that a cycle C is in \mathcal{B} if and only if $|E(C) \cap \Sigma|$ is even, and we do this by induction on the number of edges in $C \setminus E(T)$.

If all but one edge e of C is contained in T , then the result holds by definition of Σ . Suppose $|E(C) \setminus E(T)| = n \geq 2$, and the result holds for all cycles having less than n edges not in T . Choose a minimal path P in $T \setminus E(C)$ linking two vertices x, y in $V(C)$ (such a path exists since C has at least two edges not in T : say $e = uv, f \in C \setminus T$; the $u-v$ path in T avoids f and so at some vertex leaves C and then at some vertex returns to C). Cycle C is the union of two internally disjoint $x-y$ paths P_1, P_2 and together P, P_1, P_2 form a theta subgraph of G . Let $C_1 = P_1 \cup P$ and $C_2 = P_2 \cup P$. Since (G, \mathcal{B}) has no contrabalanced theta, the cycle C is unbalanced if and only if exactly one of C_1 and C_2 is unbalanced. However,

by induction (each of C_1 and C_2 has fewer edges not in T), this holds if and only if $|E(C_1) \cap \Sigma|$ and $|E(C_2) \cap \Sigma|$ have different parity. This is equivalent to $|E(C) \cap \Sigma|$ being odd, thus completing the argument. \square

Balancing vertices. If $\Omega = (G, \mathcal{B})$ is a biased graph and $v \in V(G)$ we let $\Omega - v$ denote the biased graph $(G - v, \mathcal{B}')$ where \mathcal{B}' consists of all cycles in \mathcal{B} which do not contain v . A vertex v in a biased graph Ω is *balancing* if $\Omega - v$ is balanced. When Ω has a balancing vertex, the biases of its cycles have a simple structure. We denote the set of links incident with a vertex v in a graph by $\delta(v)$.

Observation 1.7. *Let (G, \mathcal{B}) be a biased graph and suppose u is a balancing vertex in (G, \mathcal{B}) . Let $\delta(u) = \{e_1, \dots, e_k\}$. For each pair of edges e_i, e_j ($1 \leq i < j \leq k$), either all cycles containing e_i and e_j are balanced or all cycles containing e_i and e_j are unbalanced.*

Proof. Fix i, j , and consider two cycles C and C' containing e_i and e_j . Let $e_i = ux_i$ and $e_j = ux_j$. Write $C = ue_ix_iPx_je_ju$ and $C' = ue_ix_iP'x_je_ju$. Path P may be transformed into P' by a sequence of reroutings, $P = P_0, P_1, \dots, P_l = P'$ in $G - u$. Since u is balancing, each rerouting is along a balanced cycle. Hence by Lemma 1.5, at each step $m \in \{1, \dots, l\}$, the cycles $ue_ix_iP_{m-1}x_je_ju$ and $ue_ix_iP_mx_je_ju$ have the same bias. \square

The above fact prompts the introduction of a relation on $\delta(v)$ for a balancing vertex v . Namely, we define \sim on $\delta(v)$ by the rule that $e, f \in \delta(v)$ satisfy $e \sim f$ if either $e = f$ or there exists a balanced cycle containing both e and f . Clearly \sim is reflexive and symmetric. The relation \sim is also transitive: Suppose $e_1 \sim e_2$ and $e_2 \sim e_3$ and let $e_i = vx_i$ for $1 \leq i \leq 3$. Since there is a balanced cycle containing x_1vx_2 and a balanced cycle containing x_2vx_3 , there is an x_1 - x_2 path avoiding v and an x_2 - x_3 path avoiding v . Hence there is an x_1 - x_3 path P avoiding v and a P - x_2 path Q avoiding v . Let $P \cap Q = \{y\}$. Together, v, e_1, e_2, e_3, P , and Q form a theta subgraph of G . By Observation 1.7, the cycle of this theta containing e_1, e_2 and the cycle containing e_2, e_3 are both balanced. It follows that the cycle of this theta containing e_1, e_3 is also balanced, so $e_1 \sim e_3$. We summarize this important property below.

Observation 1.8. *If v is a balancing vertex of Ω , there exists an equivalence relation \sim on $\delta(v)$ so that a cycle C of Ω containing v is balanced if and only if it contains two edges from the same equivalence class.*

We call the \sim classes of $\delta(v)$ its *b-classes*.

k -signed graphs. These are generalisations of signed graphs which we use to work with biased graphs with balancing vertices and other related biased graphs. A *k -signed graph* consists of a graph G together with a collection $\Sigma = \{\Sigma_1, \dots, \Sigma_k\}$ of

subsets of $E(G)$, which we again call its *signature*. A k -signed graph gives rise to a biased graph (G, \mathcal{B}_Σ) in which a cycle $C \in \mathcal{B}_\Sigma$ if and only if $|E(C) \cap \Sigma_i|$ is even for every $1 \leq i \leq k$. Again it is straightforward to verify that Σ satisfies the theta property. We say that an arbitrary biased graph (G, \mathcal{B}) is a k -signed graph if there exists a collection Σ so that $\mathcal{B}_\Sigma = \mathcal{B}$. A 1-signed graph is a signed graph. The reader familiar with group-labelled graphs will note that signed graphs are group-labelled graphs where the associated group is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and our k -signed graphs are group-labelled by \mathbb{Z}_2^k .

Observation 1.9. *Let (G, \mathcal{B}) be a biased graph with a balancing vertex v after deleting its set U of unbalanced loops. Let $\{\Sigma_1, \dots, \Sigma_k\}$ be the partition of $\delta(v)$ into b-classes in $(G, \mathcal{B}) \setminus U$, and let $\Sigma = \{U, \Sigma_1, \dots, \Sigma_k\}$. Then (G, \mathcal{B}) is a k -signed graph with $\mathcal{B}_\Sigma = \mathcal{B}_{\Sigma \setminus \Sigma_i} = \mathcal{B}$ for every $1 \leq i \leq k$.*

Proof. This follows easily from the fact that \sim is an equivalence relation in $(G, \mathcal{B}) \setminus U$. \square

Biased graph representations. In general, a frame matroid M has more than one biased graph representing M . We will encounter several situations in which non-isomorphic biased graphs represent the same frame matroid. For our purposes, we require three results on non-isomorphic biased graphs representing the same frame matroid.

(1) If H is a graph, one way to obtain an unbalanced biased graph (G, \mathcal{B}) with $F(G, \mathcal{B}) \cong M(H)$ is to *pinch* two vertices of H , as follows. Choose two distinct vertices $u, v \in V(H)$, and let G be the graph obtained from H by identifying u and v to a single vertex w . Then $\delta(w) = \delta(u) \cup \delta(v) \setminus \{e \mid e = uv\}$ (an edge with endpoints u and v becomes a loop incident to w). Let \mathcal{B} be the set of all cycles in G not meeting both $\delta(u)$ and $\delta(v)$. It is easy to see that the circuits of the two matroids agree, so $F(G, \mathcal{B}) \cong M(H)$. The biased graph (G, \mathcal{B}) obtained by pinching u and v of H is a signed graph with $\Sigma = \delta(u)$

The signed graph obtained by pinching two vertices of a graph has a balancing vertex. Conversely, if (G, \mathcal{B}) is a signed graph with a balancing vertex u , then (G, \mathcal{B}) is obtained as a pinch of a graph H , which we may describe as follows. If $|\delta(u)/\sim| > 2$, then u is a cut vertex and each block of G contains at most two b-classes, else (G, \mathcal{B}) contains a contrabalanced theta, contradicting Proposition 1.6. Hence each block of G contains edges in at most two b-classes of $\delta(u)$. By Observation 1.9 then, \mathcal{B} is realised in each block G_i ($i \in \{1, \dots, n\}$) of G by a single set $\Sigma_i \subseteq \delta(u)$. Clearly, taking $\Sigma = \bigcup_i \Sigma_i$ yields $\mathcal{B}_\Sigma = \mathcal{B}$. Let H be the graph obtained from G by *splitting* vertex u ; that is, replace u with two vertices, u' and u'' , put all edges in Σ incident to u' , and all edges in $\delta(u) \setminus \Sigma$ incident to u'' ; put unbalanced loops as $u'u''$ edges and leave balanced loops as balanced loop incident to either u' or u'' . It is easily verified that $M(H) \cong F(G, \mathcal{B})$:

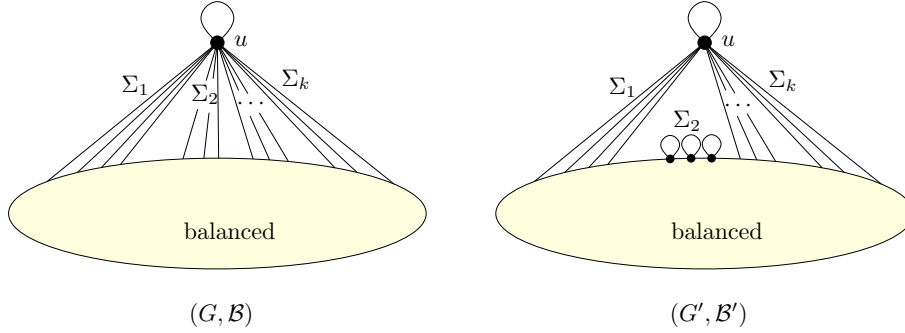


Figure 1: $F(G, \mathcal{B}) \cong F(G', \mathcal{B}')$

Proposition 1.10. *Let (G, \mathcal{B}) be a signed graph with a balancing vertex u . If H is obtained from (G, \mathcal{B}) by splitting u , then $M(H) \cong F(G, \mathcal{B})$.*

(2) If (G, \mathcal{B}) is a biased graph with a balancing vertex u , then the following *roll-up* operation produces another biased graph with frame matroid isomorphic to $F(G, \mathcal{B})$. Let $\Sigma_j = \{e_1, \dots, e_l\}$ be the set of edges of one of the b-classes in $\delta(u)$. Let (G', \mathcal{B}') be the biased graph obtained from (G, \mathcal{B}) by replacing each edge $e_i = uv_i \in \Sigma_j$ with an unbalanced loop incident to its endpoint v_i (see Figure 1). It is straightforward to check that $F(G, \mathcal{B})$ and $F(G', \mathcal{B}')$ have the same set of circuits. We say biased graph (G', \mathcal{B}') is obtained from (G, \mathcal{B}) by the *roll-up* of b-class Σ_j of $\delta(u)$. This operation may also be performed in reverse: Let (G', \mathcal{B}') be a biased graph in which u is a balancing vertex after deleting unbalanced loops, and let $\Sigma_1, \dots, \Sigma_k$ be the b-classes of $\delta(u)$ after deleting all unbalanced loops. Let (G, \mathcal{B}) be the biased graph obtained from (G', \mathcal{B}') by replacing each unbalanced loop with an edge between u and its original end, putting $\Sigma_{k+1} = \{e \mid e \text{ is an unbalanced loop in } (G', \mathcal{B}')\}$, setting $\Sigma = \{\Sigma_1, \dots, \Sigma_{k+1}\}$, and taking $\mathcal{B} = \mathcal{B}_\Sigma$. Then $F(G, \mathcal{B}) \cong F(G', \mathcal{B}')$, and we say (G, \mathcal{B}) is obtained from (G', \mathcal{B}') by *unrolling* the unbalanced loops of (G', \mathcal{B}') . Hence if M is a frame matroid represented by a biased graph with a balancing vertex u after deleting unbalanced loops, then every biased graph obtained by unrolling unbalanced loops, then rolling up a b-class of $\delta(u)$, also represents M . Observe also that if H is a graph and $v \in V(H)$, then the biased graph (G, \mathcal{B}) obtained by rolling up all edges in $\delta(v)$ has $F(G, \mathcal{B}) \cong M(H)$.

(3) The operations of pinching and splitting, and of rolling up the edges of a b-class or unrolling unbalanced loops, are all special cases of the following *twisted flip* operation. This operation may be applied to k -signed graphs having the following structure. Let G be a graph, let $u \in V(G)$, let G_0, \dots, G_m be edge disjoint connected subgraphs of G , and let $\Sigma = \{\Sigma_1, \dots, \Sigma_k\}$ be a collection of subsets of $E(G)$ satisfying the following (see Figure 2(a)).

1. $E(G) \setminus \bigcup_{i=0}^m E(G_i)$ is empty or consists of loops at u .

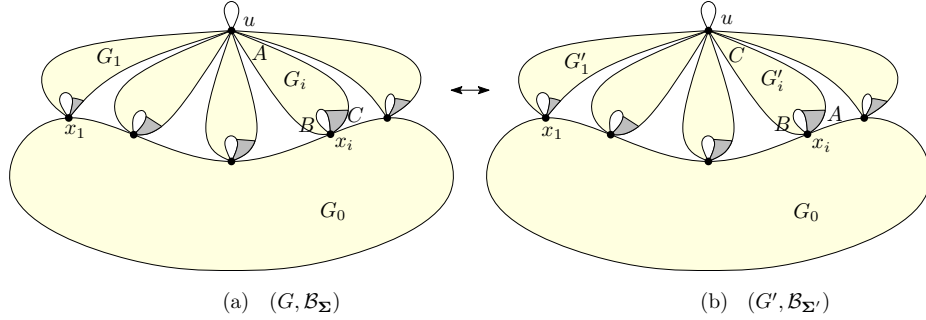


Figure 2: A twisted flip: Edges in Σ and Σ' are shaded; edges marked A in G become incident to x_i in G' and are in Σ' ; edges marked C in G become incident to u in G' .

2. $E(G_0) \cap \Sigma_i = \emptyset$ for $1 \leq i \leq k$.
3. For every $1 \leq i \leq m$ there is a vertex x_i so that $V(G_i) \cap \left(\bigcup_{j \neq i} V(G_j)\right) \subseteq \{u, x_i\}$.
4. For every $1 \leq i \leq m$ there exists a unique s_i , $1 \leq s_i \leq k$, so that $E(G_i) \cap \Sigma_j = \emptyset$ for $j \neq s_i$.
5. Every edge in $E(G_i) \cap \Sigma_{s_i}$ is incident with x_i .

Consider the resulting biased graph (G, \mathcal{B}_Σ) and its associated frame matroid $F(G, \mathcal{B}_\Sigma)$. We obtain a biased graph $(G', \mathcal{B}_{\Sigma'})$ with $F(G', \mathcal{B}_{\Sigma'}) \cong F(G, \mathcal{B}_\Sigma)$ from (G, \mathcal{B}_Σ) as follows (see Figures 2(b) and 3).

- Redefine the endpoints of each edge of the form $e=yu \notin \Sigma_{s_i}$ so that $e=yx_i$ (note that an edge $e=x_iu \notin \Sigma_{s_i}$ thus becomes a loop $e=x_ix_i$).
- Redefine the endpoints of each edge of the form $e=yx_i \in \Sigma_{s_i}$ with $y \neq u$ so that $e=yu$.
- For each $1 \leq j \leq k$, let $\Sigma'_j = \{e \mid \text{the endpoints of } e \text{ have been redefined so that } e=yx_i \text{ for some } y \in V(G_i)\} \cup \{e \mid e=x_iu \in \Sigma_j\}$. Put $\Sigma' = \{\Sigma'_1, \dots, \Sigma'_k\}$.

Theorem 1.11. *If $(G', \mathcal{B}_{\Sigma'})$ is obtained from (G, \mathcal{B}_Σ) as a twisted flip, then $F(G, \mathcal{B}_\Sigma) \cong F(G', \mathcal{B}_{\Sigma'})$.*

Proof. It is straightforward to check that $F(G, \mathcal{B}_\Sigma)$ and $F(G', \mathcal{B}_{\Sigma'})$ have the same set of circuits. \square

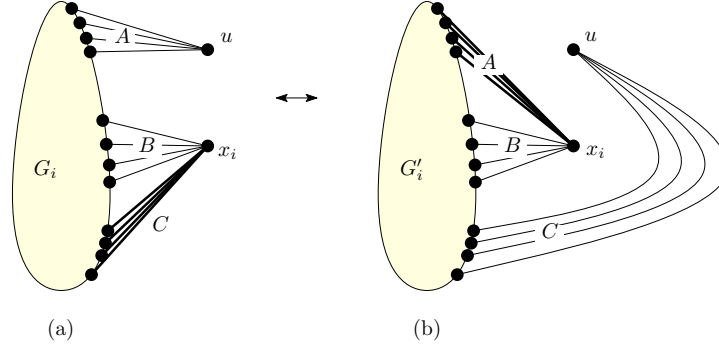


Figure 3: A twisted flip's effect on a single biased subgraph G_i . Edges contained in Σ and in Σ' are bold. In G_i (a) edges marked C are in some Σ_i , and in G'_i (b) edges marked A are then in Σ'_i .

Observe that if u is a balancing vertex in a biased graph (G, \mathcal{B}) , and $A \in \delta(u)/\sim$, then applying Observation 1.9 yields a signature $\Sigma \subseteq E(G)$ so $\mathcal{B} = \mathcal{B}_\Sigma$, with the property that A is disjoint from the members of Σ . Then a twisted flip operation on (G, \mathcal{B}_Σ) is the operation of rolling up b-class A . A pinch operation is obtained as a twisted flip by taking $G = G_1$ and $\Sigma = \emptyset$; then (G, \mathcal{B}) is balanced, and the biased graph (G', \mathcal{B}') given by a twisted flip is that obtained by pinching the vertices v and x_1 . Additionally, the special case when each $\delta_{G_i}(x_i) \cap \Sigma_{s_i} = \emptyset$ and there is no unbalanced loop incident to u is the curling operation in [1].

2 2-sums of frame matroids and matroidals

In this section we provide necessary and sufficient conditions for a 2-sum of two frame matroids to be frame, Theorem 2.2 below.

The 2-sum of two matroids M_1 and M_2 on elements $e_1 \in E(M_1)$ and $e_2 \in E(M_2)$, denoted $M_1 \overset{e_1}{\oplus}_2 \overset{e_2}{\oplus} M_2$, is the matroid on ground set $(E(M_1) \cup E(M_2)) \setminus \{e_1, e_2\}$ with circuits: the circuits of M_i avoiding e_i for $i = 1, 2$, together with $\{(C_1 \cup C_2) \setminus \{e_1, e_2\} \mid C_i \text{ is a circuit of } M_i \text{ containing } e_i \text{ for } i = 1, 2\}$. The following result (independently of Bixby, Cunningham, and Seymour) is fundamental.

Theorem 2.1 ([8], Theorem 8.3.1). *A connected matroid M is not 3-connected if and only if there are matroids M_1, M_2 , each of which is a proper minor of M , such that M is a 2-sum of M_1 and M_2 .*

If M is a matroid whose automorphism group is transitive on $E(M)$, then we write simply $M \overset{f}{\oplus}_2 N$ to indicate the 2-sum of M and N taken on some element $e \in E(M)$ and element $f \in E(N)$; if also N has transitive automorphism group we may simply write $M \oplus_2 N$.

Matroidals. A *matroidal* is a pair (M, L) consisting of a matroid M together with a distinguished subset L of its elements. A matroidal $\mathcal{M}=(M, L)$ is *frame* if there is a biased graph Ω with $M = F(\Omega)$ in which every element in L is an unbalanced loop. We say a biased graph in which all elements in $L \subseteq E(\Omega)$ are unbalanced loops is *L-biased*. Thus $\mathcal{M}=(M, L)$ is a frame matroidal if and only if there exists an L -biased graph Ω with $F(\Omega) = M$. In this case we say Ω *represents* \mathcal{M} .

Observe that, as long as M is simple, this is equivalent to asking that there be a frame for M containing L . To see this, recall the construction given on page 1 of a biased graph representing a matroid M with frame B . Though it is not required that the frame B be disjoint from E , the construction assumes $B \cap E = \emptyset$. We can do away with this assumption as follows. Suppose $B \cap E = F$. Construct (G, \mathcal{B}) with edge set $E \setminus F$ as before. Now add an unbalanced loop incident to each vertex of G , and let each element of the frame be represented by the new loop incident to its vertex. Thus we obtain a frame extension N without any added parallel elements in which all elements in the frame are unbalanced loops in the biased graph representing N . Conversely, as long as M is simple, given a biased graph Ω representing M , the set of unbalanced loops of Ω is contained in a frame for M — namely, after adding an unbalanced loop at each vertex not already having one, the basis consisting of the set of unbalanced loops.

The main result of this section says that a 2-sum of two non-graphic frame matroids is frame if and only if each of the summands has a frame containing the element upon which the 2-sum is taken.

Theorem 2.2. *Let M_1, M_2 be connected matroids and for $i = 1, 2$ let $e_i \in E(M_i)$. The matroid $M_1 \oplus^{e_1}_{e_2} M_2$ is frame if and only if one of the following holds.*

1. *One of M_1 or M_2 is graphic and the other is frame.*
2. *Both matroidals $(M_1, \{e_1\})$ and $(M_2, \{e_2\})$ are frame.*

We prove a more general statement than Theorem 2.2, giving necessary and sufficient conditions for a 2-sum of two frame matroidals to be frame. This more general result will be required in Section 3. The statement and its proof will be given after the following necessary preliminaries.

2.1 2-summing biased graphs

Let Ω_1, Ω_2 be biased graphs and let $e_i \in E(\Omega_i)$ for $i = 1, 2$. There are two ways in which we may perform a biased graphical 2-sum operation on Ω_1 and Ω_2 to obtain a biased graph representing the 2-sum $F(\Omega_1) \oplus^{e_1}_{e_2} F(\Omega_2)$.

1. Suppose that e_i is an unbalanced loop in Ω_i incident with vertex v_i , for $i \in \{1, 2\}$. The *loop-sum* of Ω_1 and Ω_2 on e_1 and e_2 is the biased graph obtained from the disjoint union of $\Omega_1 - e_1$ and $\Omega_2 - e_2$ by identifying vertices v_1 and

v_2 . Every cycle in the loop-sum is contained in one of Ω_1 or Ω_2 ; its bias is defined accordingly.

2. Suppose that Ω_1 is balanced, and that e_i is a link in Ω_i incident with vertices u_i, v_i , for $i \in \{1, 2\}$. The *link-sum* of Ω_1 and Ω_2 on e_1 and e_2 is the biased graph obtained from the disjoint union of $\Omega_1 - e_1$ and $\Omega_2 - e_2$ by identifying u_1 with u_2 and v_1 with v_2 . A cycle in the link-sum is balanced if it is either a balanced cycle in Ω_1 or Ω_2 or if it may be written as a union $(C_1 \setminus e_1) \cup (C_2 \setminus e_2)$ where for $i \in \{1, 2\}$, C_i is a balanced cycle in Ω_i containing e_i . (It is straightforward to verify that the theta rule is satisfied by this construction.)

Proposition 2.3. *Let Ω_1, Ω_2 be biased graphs and let $e_i \in E(\Omega_i)$ for $i \in \{1, 2\}$. If Ω is a loop-sum or link-sum of Ω_1 and Ω_2 on e_1 and e_2 , then $F(\Omega) = F(\Omega_2)^{e_1 \oplus_2^{e_2}} F(\Omega_1)$.*

Proof. It is easily checked that for both the loop-sum and link-sum, the circuits of $F(\Omega)$ and of $F(\Omega_1)^{e_1 \oplus_2^{e_2}} F(\Omega_2)$ coincide, regardless of the choice of pairs of endpoints of e_1 and e_2 that are identified in the link-sum. \square

2.2 Decomposing along a 2-separation

By Theorem 2.1, a matroid M of connectivity 2 decomposes into two of its proper minors such that M is a 2-sum of these smaller matroids. If M is frame, then every minor of M is frame, and we would like to be able to express the 2-sum in terms of a loop-sum or link-sum of two biased graphs representing these minors. This motivates the following definitions. Let M be a connected frame matroid on E and let Ω be a biased graph representing M . A 2-separation (A, B) of M is a *biseparation* of Ω . There are four types of biseparations that play key roles. Define a biseparation to be type 1, 2(a), 2(b), 3(a), 3(b), or 4, respectively, if it appears as in Figure 4, where each component of $\Omega[A]$ and $\Omega[B]$ is connected; components of each side of the separation marked “b” are balanced, those marked “u” are unbalanced. We refer to a biseparation of type 2(a) or 2(b) as type 2, and a biseparation of type 3(a) or 3(b) as type 3.

Proposition 2.4. *Let M be a connected frame matroid such that $M = M_1^{e_1 \oplus_2^{e_2}} M_2$ for two matroids M_1, M_2 . Let Ω be a biased graph representing M , and let $E(M_i) \setminus \{e_i\} = E_i$ for $i \in \{1, 2\}$. If (E_1, E_2) is type 1 (resp. type 2), then there exist biased graphs Ω_i with $E(\Omega_i) = E(M_i)$, $i \in \{1, 2\}$, such that Ω is the loop-sum (resp. link-sum) of Ω_1 and Ω_2 on e_1 and e_2 .*

Proof. If (E_1, E_2) is type 1, then for $i \in \{1, 2\}$ let Ω_i be the biased graph obtained from Ω by replacing $\Omega[E_{i+1}]$ with an unbalanced loop e_i incident to the vertex in $V(E_1) \cap V(E_2)$ (adding indices modulo 2). Then Ω is the loop-sum of Ω_1 and Ω_2 on e_1 and e_2 .

Now suppose (E_1, E_2) is type 2. Let $V(E_1) \cap V(E_2) = \{x, y\}$, and assume without loss of generality that $\Omega[E_1]$ is balanced while $\Omega[E_2]$ is unbalanced. For

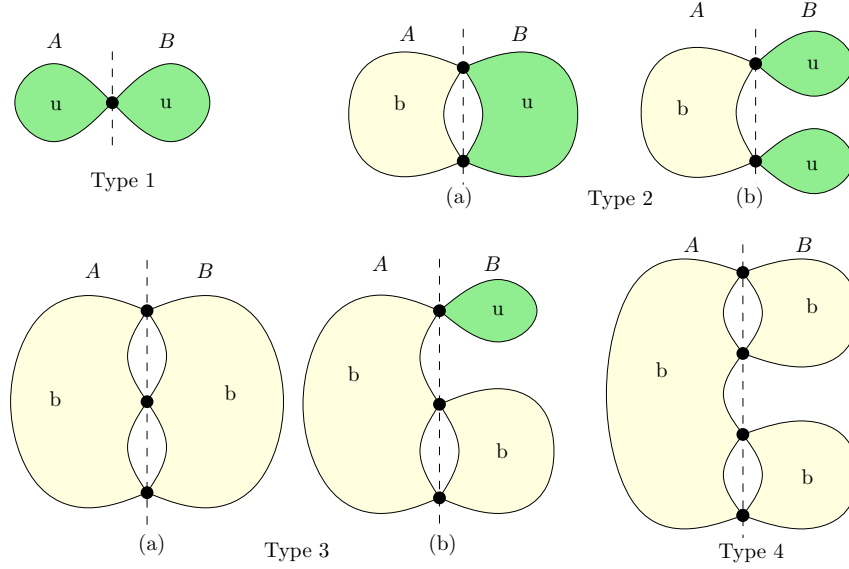


Figure 4: Four types of biseparations.

$i \in \{1, 2\}$ let Ω_i be the biased graph obtained from Ω by replacing $\Omega[E_{i+1}]$ with a link $e_i = xy$ and defining bias as follows: Ω_1 is balanced, while the balanced cycles of Ω_2 are precisely those not containing e_2 that are balanced in Ω together with those cycles containing e_2 for which replacing e_2 with an x - y path in $\Omega[E_1]$ yields a balanced cycle in Ω (Lemma 1.5 guarantees that this collection is well-defined). Then Ω is the link-sum of Ω_1 and Ω_2 on e_1 and e_2 . \square

2.2.1 Taming biseparations

In light of Proposition 2.4, we want to show that for every 2-separation of a frame matroid M , there exists a biased graph representing M for which the corresponding biseparation is type 1 or 2. We first show that there is always such a representation in which the biseparation is type 1, 2, or 3. In preparation for the more general form of Theorem 2.2 we wish to prove, we now consider matroidals. We say a matroidal $\mathcal{M}=(M, L)$ is connected if M is connected.

Lemma 2.5. *Let $\mathcal{M}=(M, L)$ be a connected frame matroidal. For every 2-separation (A, B) of M , there exists an L -biased representation of \mathcal{M} for which (A, B) is type 1, 2, or 3.*

Proof. Choose an L -biased representation Ω of (M, L) for which Ω is not balanced (any balanced representation can be turned into an unbalanced one by a pinch or roll-up operation, so this is always possible). Let $S = V(A) \cap V(B)$ in Ω . Let $\{A_1, \dots, A_h\}$ be the partition of A and $\{B_1, \dots, B_k\}$ the partition of B so that every

$\Omega[A_i]$ is a component of the biased graph $\Omega[A]$ and every $\Omega[B_j]$ is a component of the biased graph $\Omega[B]$. Call the graphs $\Omega[A_1], \dots, \Omega[A_h], \Omega[B_1], \dots, \Omega[B_k]$ *parts*. For every $1 \leq i \leq h$ (resp. $1 \leq j \leq k$) let $\delta_A^i = 1$ ($\delta_B^j = 1$) if $\Omega[A_i]$ is balanced ($\Omega[B_j]$ is balanced) and $\delta_A^i = 0$ ($\delta_B^j = 0$) otherwise. Then $\lambda_M(A, B) = 2 = 1 + |S| - \sum_{i=1}^h \delta_A^i - \sum_{j=1}^k \delta_B^j$. Since each vertex in S is in exactly one $\Omega[A_i]$ and exactly one $\Omega[B_j]$, doubling both sides of this equation and rearranging, we obtain

$$2 = \sum_{i=1}^h (|S \cap V(A_i)| - 2\delta_A^i) + \sum_{j=1}^k (|S \cap V(B_j)| - 2\delta_B^j).$$

If a part is balanced, it must contain at least two vertices in S (else M is not connected by the discussion in Section 1.1), so every term in the sums on the right hand side of the above equation is nonnegative. In particular, letting t be the number of vertices in S contained in a part, a balanced part will contribute $t - 2$ to the sum, and an unbalanced part will contribute t . Call a part *neutral* if it is balanced and contains exactly two vertices in S . Since the total sum is two, the possibilities for the parts of $\Omega[A]$ and $\Omega[B]$ are:

- (a) two unbalanced parts each with one vertex in S and all other parts neutral,
- (b) one unbalanced part with two vertices in S and all other parts neutral,
- (c) one balanced part with three vertices in S , one unbalanced part with one vertex in S , and all other parts neutral,
- (d) two balanced parts with three vertices in S and all other parts neutral, or
- (e) one balanced part with four vertices in S and all other parts neutral.

These possibilities are illustrated in Figure 5.

Observe that every component of $M \setminus B$ (resp. $M \setminus A$) is contained in some part $\Omega[A_i]$ ($\Omega[B_j]$), and every part of $\Omega[A]$ (resp. $\Omega[B]$) is a union of components of $M \setminus B$ (resp. $M \setminus A$). Hence every circuit of M is either contained in a single part, or traverses every part. It is an elementary property of 2-separations that if A_1, \dots, A_l and B_1, \dots, B_m are the components of $M \setminus B$ and $M \setminus A$ respectively, and (X, Y) is any partition of $A_1, \dots, A_l, B_1, \dots, B_m$, then $(\bigcup X, \bigcup Y)$ is a 2-separation of M (this can be verified by straightforward rank calculations). Hence if $\Omega[D]$ is a neutral part, (D, D^c) is a 2-separation of M . Since $\Omega[D]$ is balanced and connected, the biseparation (D, D^c) of Ω is type 2.

Suppose there are exactly t neutral parts. Repeatedly applying Proposition 2.4, we obtain a biased graph Ω' with links e'_1, \dots, e'_t together with balanced biased graphs $\Omega_1, \dots, \Omega_t$ each with a distinguished edge $e_i \in E(\Omega_i)$ so that Ω is obtained as a repeated link-sum of Ω' with each Ω_i on edges e_i and e'_i . It follows from the fact that every circuit of M is either contained in a single part or traverses every part

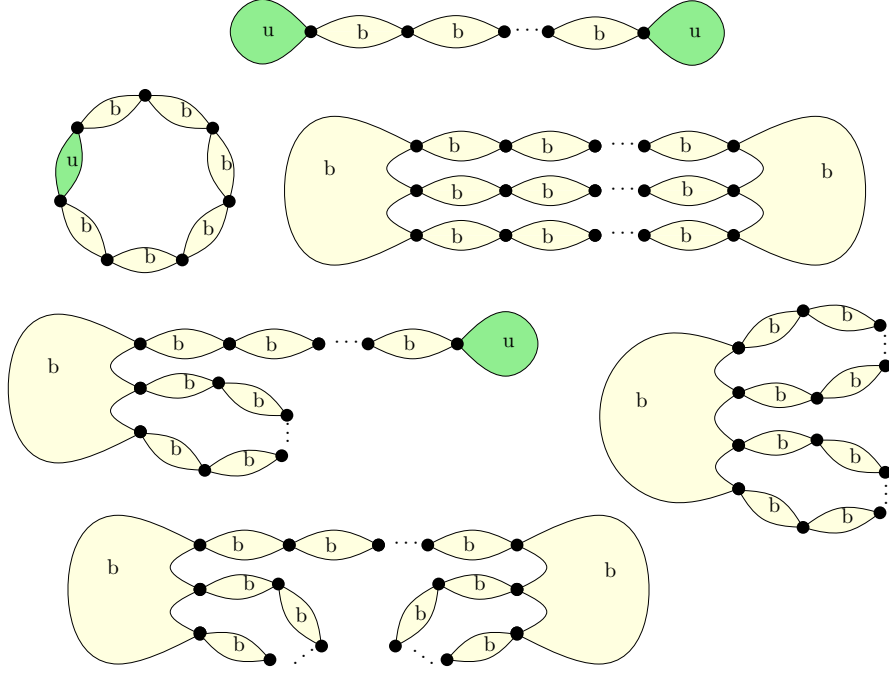


Figure 5: The possible decompositions of Ω into the parts of $\Omega[A]$ and $\Omega[B]$.

that elements e'_1, \dots, e'_t are all in series in $F(\Omega')$. We use this fact to find another biased graph representing M in which the biseparation (A, B) is type 1, 2, or 3. First, in Ω' contract edges e'_1, \dots, e'_{t-1} : let $\Omega'' = \Omega' / \{e'_1, \dots, e'_{t-1}\}$. Now subdivide link e'_t to form a path P with edge set $\{e'_1, \dots, e'_t\}$ to obtain a new biased graph Ψ , in which a cycle containing P is balanced if and only if the corresponding cycle in Ω'' containing e'_t is balanced. Since elements e'_1, \dots, e'_t are in series in $F(\Omega')$, $F(\Psi) \cong F(\Omega')$. For the same reason, any biased graph Ψ' obtained from Ψ by permuting the order in which edges e'_1, \dots, e'_t occur in P has $F(\Psi') \cong F(\Psi)$. Let Φ' be the biased graph obtained from Ψ by arranging the edges of P in an order so that an initial segment of the path has all of the edges e'_i whose corresponding neutral parts of Ω are in A , followed by the edges e'_i whose corresponding neutral parts are in B . Now let Φ be the biased graph obtained by repeatedly link-summing each Ω_i on edge e'_i , $i \in \{1, \dots, t\}$. Then $F(\Phi) \cong F(\Omega)$. Since every unbalanced loop in Ω remains an unbalanced loop in Φ , Φ is an L -biased representation of \mathcal{M} . Since at least one of $\Phi[A]$ or $\Phi[B]$ is connected, and $\Phi[A]$ and $\Phi[B]$ meet in at most three vertices, in Φ biseparation (A, B) is type 1, 2, or 3. \square

2.2.2 Taming type 3

We now do away with type 3 biseparations.

Theorem 2.6. *Let $\mathcal{M}=(M, L)$ be a connected frame matroidal. For every 2-separation (A, B) of M , there exists an L -biased representation of \mathcal{M} for which (A, B) is type 1 or 2.*

Proof. By Lemma 2.5 we may choose an L -biased graph Ω representing M in which biseparation (A, B) is type 1, 2, or 3. Suppose it is type 3. Let $\{x, y, z\} = V(A) \cap V(B)$ in Ω . We first claim that all cycles crossing (A, B) through the same pair of vertices $\{x, y\}$, $\{y, z\}$, or $\{z, x\}$ have the same bias. To see this, let C and C' be two cycles crossing (A, B) at $\{x, y\}$. We may assume without loss of generality that $\delta(z) \cap C \subseteq A$. Let $C \cap A = P$ and $C \cap B = Q$, and let $C' \cap A = P'$ and $C' \cap B = Q'$. By Observation 1.4, P may be transformed to P' by a sequence of reroutings in $P \cup P'$. Since every rerouting in this sequence is along a balanced cycle, by Lemma 1.5, C and $P' \cup Q$ have the same bias. Similarly, Q may be transformed into Q' via a sequence of reroutings along balanced cycles in $Q \cup Q'$, so $P' \cup Q$ and $P' \cup Q' = C'$ have the same bias. *I.e.*, C and C' are of the same bias.

There are three types of cycles crossing the 2-separation: those crossing at $\{x, y\}$, those crossing at $\{x, z\}$, and those crossing at $\{y, z\}$; by the claim, all cycles of the same type have the same bias. Let us denote the sets of these cycles by \mathcal{C}_{xy} , \mathcal{C}_{xz} and \mathcal{C}_{yz} , respectively.

We claim that at least one of these sets contains an unbalanced cycle. For suppose the contrary. If the biseparation of Ω is type 3(a), then Ω is balanced with $|V(A) \cap V(B)| = 3$; but then (A, B) is not a 2-separation of $F(\Omega)$, a contradiction. If the biseparation is type 3(b), then M is not connected, a contradiction.

Suppose first that just one of our sets of cycles, say \mathcal{C}_{xy} , contains an unbalanced cycle C . Suppose further that in one of $\Omega[A]$ or $\Omega[B]$ there is a z - C path P avoiding x and that in the other side there is a z - C path Q avoiding y . Then $C \cup P \cup Q$ is a theta subgraph of Ω containing exactly two balanced cycles, a contradiction. So no such pair of paths exist. Hence either:

1. both $\Omega[A]$ and $\Omega[B]$ contain a z - C path, but either every z - C path in both meets x or every z - C path in both meets y , or,
2. one of $\Omega[A]$ or $\Omega[B]$ has no z - C path.

In case 1, either x or y is a cut vertex of Ω , and we find that $F(\Omega)$ is not connected, a contradiction. Hence we have case 2. Suppose without loss of generality that $\Omega[B]$ does not contain a z - C path. We have a biseparation of type 3(b). Let us denote by B_1 the balanced component and by B_2 the unbalanced component of biased graph $\Omega[B]$. Let Φ be the biased graph obtained as follows. Detach B_2 from $\Omega[A]$, and form a signed graph (G, \mathcal{B}_Σ) from $\Omega[A]$ by identifying vertices x and y , and setting $\Sigma = \delta(y) \cap A$. Now identify vertex x in B_1 with vertex z in (G, \mathcal{B}_Σ) , and identify vertex y in B_1 with vertex z in B_2 (Figure 6). Assign biases to cycles in Φ in $\Phi[A]$ according to their bias in (G, \mathcal{B}_Σ) and in $\Phi[B]$ according to their bias in Ω . It is straightforward to verify that the circuits of $F(\Phi)$ and $F(\Omega)$ coincide, so $F(\Phi) \cong M$.

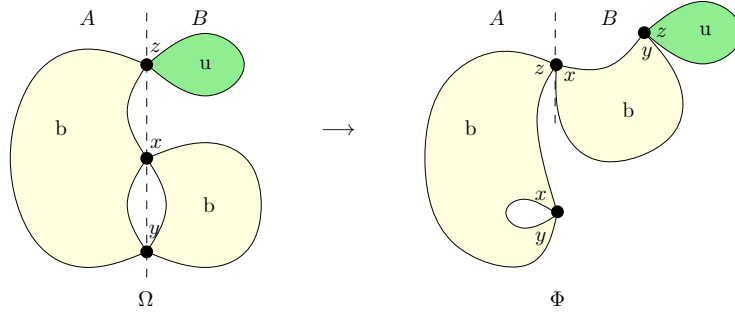


Figure 6: Finding a representation in which the biseparation is type 1.

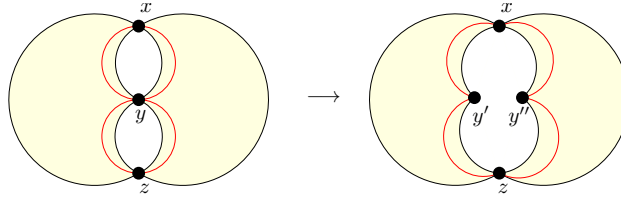


Figure 7: If just \mathcal{C}_{xy} and \mathcal{C}_{yz} contain unbalanced cycles, then $F(\Omega)$ is graphic.

The biseparation (A, B) in Φ is type 1, and since edges representing elements in L remain unbalanced loops in Φ , Φ is an L -bias representation of M as required.

So now assume that at least two of the three sets \mathcal{C}_{xy} , \mathcal{C}_{yz} and \mathcal{C}_{xz} contain an unbalanced cycle. Then our biseparation is type 3(a). If just two of these sets contain an unbalanced cycle — say \mathcal{C}_{xz} does not — then M is graphic, represented by the graph obtained from Ω by splitting vertex y (Figure 7). Now pinching vertices x and z yields an L -biased graph representing M in which biseparation (A, B) is type 1.

The remaining case is that all three of \mathcal{C}_{xy} , \mathcal{C}_{xz} , and \mathcal{C}_{yz} contain unbalanced cycles, so every cycle crossing (A, B) is unbalanced. In this case every circuit of M contained in A or B is a balanced cycle and every circuit meeting both A and B is either a pair of tight handcuffs meeting at a vertex in $V(A) \cap V(B)$, or an contrabalanced theta (Figure 8). Let Ω' be the signed graph obtained from Ω as follows. Split vertices y and z , replacing y with two new vertices y' and y'' , putting

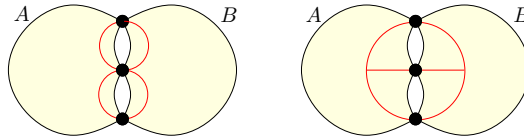


Figure 8: Circuits of $F(\Omega)$ meeting both sides of the 2-separation.

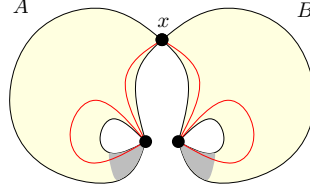


Figure 9: A representation in which the biseparation is type 1

all edges $uy \in A$ incident with y' and all edges $vy \in B$ incident with y'' and similarly replacing z with two new vertices z' and z'' , putting all edges $uz \in A$ incident with z' and all edges $vz \in B$ incident with z'' . Now identify vertices y' and z' and identify vertices y'' and z'' , and put the edges in $\delta(z) \cap A$ and in $\delta(z) \cap B$ in Σ (Figure 9). It is easily checked that a subset $C \subseteq E$ is a circuit in $F(\Omega)$ if and only if C is a circuit in $F(\Omega')$, so $F(\Omega') \cong F(\Omega)$. Since in this case L is empty, Ω' is an L -biased graph representing M , as required. \square

2.3 Proof of Theorem 2.2

With this we are ready to prove the main result of this section.

Lemma 2.7. *Let $\mathcal{M}_1=(M_1, L_1)$ and $\mathcal{M}_2=(M_2, L_2)$ be connected frame matroids on E_1, E_2 , respectively. If for $i = 1, 2$, $e_i \in E_i \setminus L_i$, then $(M_1 \overset{e_1}{\oplus} M_2, L_1 \cup L_2)$ is frame if and only if one of the following holds.*

1. $L_i = \emptyset$ and M_i is graphic for one of $i = 1$ or $i = 2$.
2. $(M_i, L_i \cup \{e_i\})$ is frame for both $i = 1, 2$.

Proof. The “if” direction follows immediately from Proposition 2.3. Conversely, consider a frame matroidal resulting from a 2-sum, $(M_1 \overset{e_1}{\oplus} M_2, L_1 \cup L_2)$. By Theorem 2.6 there is a $(L_1 \cup L_2)$ -biased representation Ω of the 2-sum in which the biseparation $(E_1 \setminus e_1, E_2 \setminus e_2)$ is type 1 or 2. By Proposition 2.4, there are biased graphs Ω_1 on E_1 and Ω_2 on E_2 such that Ω is a link- or loop-sum on e_1 and e_2 . If Ω is a link-sum, then 1 holds. If Ω is a loop-sum, then both Ω_i are $(L_i \cup \{e_i\})$ -biased representations of M_i , so both matroids $(M_i, L_i \cup \{e_i\})$ are frame ($i \in \{1, 2\}$). \square

Lemma 2.7 immediately implies Theorem 2.2.

Proof of Theorem 2.2. Apply Lemma 2.7 with $L_1 = L_2 = \emptyset$. \square

3 Excluded minors

In this section we use Theorem 2.2 to construct a family \mathcal{E}_0 of 9 excluded minors with connectivity 2. We then show that any excluded minor of connectivity 2 that is not in \mathcal{E}_0 has a special structure.

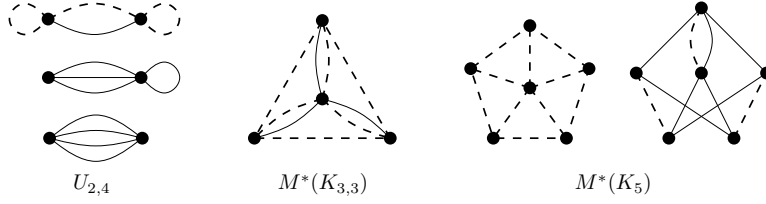


Figure 10: The biased graphs representing excluded minors for the class of graphic matroids. Those with dashed edges are signed graphs, with signatures given by dashed edges. The other two biased graphs, representing $U_{2,4}$, have all cycles unbalanced.

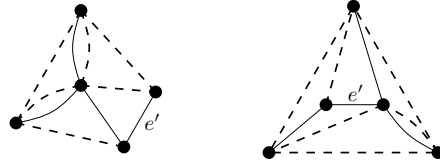


Figure 11: The biased graphs representing $M^*(K'_{3,3})$; both are signed graphs with signature the dashed edges.

3.1 The excluded minors \mathcal{E}_0

The graph obtained from $K_{3,3}$ by adding an edge e' linking two non-adjacent vertices is denoted $K'_{3,3}$; we also denote the corresponding element of $M^*(K'_{3,3})$ by e' . Let

$$\begin{aligned} \mathcal{E}_0 = & \{U_{2,4} \oplus_2 M^*(H) \mid H \in \{K_5, K_{3,3}, K'_{3,3}\}\} \\ & \cup \{M^*(H_1) \oplus_2 M^*(H_2) \mid H_1, H_2 \in \{K_5, K_{3,3}, K'_{3,3}\}\}, \end{aligned}$$

where the 2-sum is taken on e' whenever H , H_1 or H_2 is $K'_{3,3}$.

There are three biased graphs representing $U_{2,4}$, two biased graphs representing $M^*(K_5)$, and just one biased graph representation of $M^*(K_{3,3})$ [11]. These are shown in Figure 10. There are two biased graphs representing $M^*(K'_{3,3})$, shown in Figure 11.

Theorem 3.1. *The matroids in \mathcal{E}_0 are excluded minors for the class of frame matroids.*

Proof. Let $M_1 \oplus_2 M_2 \in \mathcal{E}_0$, with M_1 one of $U_{2,4}$, $M^*(K_5)$, $M^*(K_{3,3})$, or $M^*(K'_{3,3})$ and M_2 one of $M^*(K_5)$, $M^*(K_{3,3})$, or $M^*(K'_{3,3})$. Since neither M_1 nor M_2 is graphic and M_2 has no representation with a loop, by Theorem 2.2 $M_1 \oplus_2 M_2$ is not frame. Since every proper minor of $U_{2,4}$, $M^*(K_5)$, and $M^*(K_{3,3})$ is graphic, and for every $e \neq e'$, both $M^*(K'_{3,3}) \setminus e$ and $M^*(K'_{3,3})/e$ are graphic, every proper minor of

$M_1 \oplus_2 M_2$ is a 2-sum of a graphic matroid and a frame matroid. Hence by Theorem 2.2, every proper minor of $M_1 \oplus_2 M_2$ is frame. \square

3.2 Other excluded minors of connectivity 2

We now investigate excluded minors of connectivity 2 that are not in \mathcal{E}_0 . We show that any such excluded minor has the following structure. For a matroid M and subset $L \subseteq E(M)$, the matroid obtained by taking a 2-sum of a copy of $U_{2,4}$ on each element in L is denoted $M \overset{L}{\oplus}_2 U_{2,4}$.

Theorem 3.2. *Let M be an excluded minor for the class of frame matroids. If M has connectivity 2 and is not in \mathcal{E}_0 , then $M = N \overset{L}{\oplus}_2 U_{2,4}$ for a 3-connected frame matroid N .*

We prove Theorem 3.2 via three lemmas, each of which requires some explanation.

A collection \mathcal{N} of connected matroids is *1-rounded* if it has the property that whenever a connected matroid M has a minor $N \in \mathcal{N}$, then every element $e \in E(M)$ is contained in some minor N' of M with $N' \in \mathcal{N}$. The following is a result of Seymour ([8] Section 11.3).

Theorem 3.3. *The collection $\{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3})\}$ is 1-rounded.*

We use Theorem 3.3 in the proof of the following lemma, to find a minor containing the base point on which a 2-sum is taken.

Lemma 3.4. *Let M_1, M_2 be nontrivial matroids and suppose $M_1 \overset{e_1}{\oplus}_2 \overset{e_2}{\oplus} M_2$ is an excluded minor for the class of frame matroids, for some $e_1 \in E(M_1)$ and $e_2 \in E(M_2)$. Then either $M_1 \overset{e_1}{\oplus}_2 \overset{e_2}{\oplus} M_2 \in \mathcal{E}_0$ or both M_1 and M_2 are non-binary frame matroids.*

Proof. By minimality, M_1 and M_2 are both frame. By Theorem 2.2, neither M_1 nor M_2 is graphic. Thus each contains an excluded minor for the class of graphic matroids, namely, one of $U_{2,4}, F_7, F_7^*, M^*(K_5)$, or $M^*(K_{3,3})$. By Theorem 3.3, for $i \in \{1, 2\}$, matroid M_i contains a minor N_i isomorphic to one of $U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})$, or $M^*(K'_{3,3})$ with $e_i \in E(N_i)$; we may assume that if $N_i \cong M^*(K'_{3,3})$ then e_i is edge e' . Since neither F_7 nor F_7^* are frame, neither N_1 nor N_2 is isomorphic to F_7 or F_7^* . If $N_1 \overset{e_1}{\oplus}_2 \overset{e_2}{\oplus} N_2 \in \mathcal{E}_0$, then by minimality, for $i \in \{1, 2\}$, $M_i \cong N_i$ and $M_1 \overset{e_1}{\oplus}_2 \overset{e_2}{\oplus} M_2 \cong N_1 \overset{e_1}{\oplus}_2 \overset{e_2}{\oplus} N_2$. Otherwise, $N_1 \cong N_2 \cong U_{2,4}$, so both M_1 and M_2 are non-binary. \square

Our next lemma requires two basic facts. The first is a result of Bixby; the second was proved independently by Brylawski and Seymour.

Proposition 3.5 ([8], Proposition 11.3.7). *Let M be a connected matroid having a $U_{2,4}$ minor and let $e \in E(M)$. Then M has a $U_{2,4}$ minor using e .*

Proposition 3.6 ([8], Proposition 4.3.6). *Let N be a connected minor of a connected matroid M and suppose that $e \in E(M) \setminus E(N)$. Then at least one of $M \setminus e$ and M/e is connected and has N as a minor.*

Lemma 3.7. *Let $M_1 \stackrel{e_1}{\oplus} \stackrel{e_2}{\oplus} M_2$ be an excluded minor for the class of frame matroids with both M_1 and M_2 non-binary. Then one of M_1 or M_2 is isomorphic to $U_{2,4}$.*

Proof. Suppose for a contradiction that neither M_1 nor M_2 is isomorphic to $U_{2,4}$. By Propositions 3.5 and 3.6 we may choose an element $f \in E(M_1) \setminus \{e_1\}$ so that a matroid M'_1 obtained from M_1 by either deleting or contracting f is connected and has $U_{2,4}$ as a minor. Since $M'_1 \stackrel{e_1}{\oplus} \stackrel{e_2}{\oplus} M_2$ is a minor of $M_1 \stackrel{e_1}{\oplus} \stackrel{e_2}{\oplus} M_2$, by minimality $M'_1 \stackrel{e_1}{\oplus} \stackrel{e_2}{\oplus} M_2$ is frame. By Theorem 2.2, $(M_2, \{e_2\})$ is frame. Similarly, $(M_1, \{e_1\})$ is frame. Hence by Theorem 2.2, $M_1 \stackrel{e_1}{\oplus} \stackrel{e_2}{\oplus} M_2$ is frame, a contradiction. \square

The final lemma we need to prove Theorem 3.2 tells us that in our current setting, 2-separations having one side just a 3-circuit cannot interact with each other. The complement of a subset $A \subseteq E$ is denoted A^c .

Lemma 3.8. *Let M be a connected matroid on E with $|E| \geq 6$ and assume that for every 2-separation (A, A^c) of M , one of $M[A]$ or $M[A^c]$ is a circuit of size 3. If (A, A^c) and (B, B^c) are 2-separations with both $M[A]$ and $M[B]$ a circuit of size 3, then either $A = B$ or $A \cap B = \emptyset$.*

Proof. Suppose for a contradiction that $\emptyset \neq A \cap B \neq A$. We consider two cases depending on the size of $A \cap B$. If $|A \cap B| = 1$ then let $A \cap B = \{e\}$ and consider the separation $(A \setminus \{e\}, A^c \cup \{e\})$. Since $B \setminus \{e\}$ spans e , $r(A^c) = r(A^c \cup \{e\})$. But this implies that $(A \setminus \{e\}, A^c \cup \{e\})$ is a 2-separation, a contradiction as neither side has size three.

Next suppose $|A \cap B| = 2$. Then, summing the orders of the separations $(A \cap B, A^c \cup B^c)$ and $(A \cup B, A^c \cap B^c)$, by submodularity, we have

$$\begin{aligned} \lambda_M(A \cap B, A^c \cup B^c) + \lambda_M(A \cup B, A^c \cap B^c) \\ &= r(A \cap B) + r(A^c \cup B^c) + r(A \cup B) + r(A^c \cap B^c) - 2r(M) + 2 \\ &\leq r(A) + r(A^c) + r(B) + r(B^c) - 2r(M) + 2 \\ &= \lambda_M(A, A^c) + \lambda_M(B, B^c) = 4. \end{aligned}$$

As M is connected, each of $\lambda_M(A \cap B, A^c \cup B^c)$ and $\lambda_M(A \cup B, A^c \cap B^c)$ is at least two, so this implies that $(A \cap B, A^c \cup B^c)$ is a 2-separation, again a contradiction. \square

Proof of Theorem 3.2. Let M be an excluded minor for the class frame matroids, and suppose M has connectivity 2 and $M \notin \mathcal{E}_0$. By Lemma 3.4, whenever M is written as a 2-sum, each term of the sum is non-binary, and by Lemma 3.7 one of

these terms is isomorphic to $U_{2,4}$. Hence every 2-separation (A, A^c) of M has one of $M[A]$ or $M[A^c]$ a circuit of size 3. By Lemma 3.8 the 3-circuits corresponding to these $U_{2,4}$ minors are pairwise disjoint. Therefore we may write $M = N \oplus_2^L U_{2,4}$, where N is a 3-connected matroid. \square

3.3 Excluded minors for the class of frame matroidals

Theorem 3.2 says that every excluded minor of connectivity 2 for the class of frame matroids that is not in \mathcal{E}_0 can be expressed in the form $N \oplus_2^L U_{2,4}$, where N is a 3-connected frame matroid. In this section we equate the problem of representing a matroid of this form as a biased graph to frame matroidals. We begin with the following key result.

Theorem 3.9. *Let N be a matroid and let $L \subseteq E(N)$. Then $N \oplus_2^L U_{2,4}$ is frame if and only if the matroidal (N, L) is frame.*

Proof. Let $L = \{e_1, \dots, e_k\}$ and repeatedly apply Lemma 2.7:

$$\begin{aligned}
N \oplus_2^L U_{2,4} \text{ is frame} &\iff \left((N \oplus_2^{\{e_1 \dots e_{k-1}\}} U_{2,4})^{e_k} \oplus_2 U_{2,4}, \emptyset \right) \text{ is frame} \\
&\iff \left((N \oplus_2^{\{e_1 \dots e_{k-2}\}} U_{2,4})^{e_{k-1}} \oplus_2 U_{2,4}, \{e_k\} \right) \text{ is frame} \\
&\iff \left((N \oplus_2^{\{e_1 \dots e_{k-3}\}} U_{2,4})^{e_{k-2}} \oplus_2 U_{2,4}, \{e_{k-1}, e_k\} \right) \text{ is frame} \\
&\quad \vdots \\
&\iff (N, \{e_1, \dots, e_k\}) \text{ is frame.} \quad \square
\end{aligned}$$

So that we may work directly with matroidals, we now define minors of matroidals. Any matroidal (N, K) obtained from a matroidal (M, L) by a sequence of the operations of deleting or contracting an element not in L or removing an element from L is a *minor* of (M, L) . Clearly, the class of frame matroidals is minor-closed, and so we may ask for its set of excluded minors. We have the following immediate corollary of Theorem 3.9.

Corollary 3.10. *Let N be a matroid and let $L \subseteq E(N)$. Then $N \oplus_2^L U_{2,4}$ is an excluded minor for the class of frame matroids if and only if (N, L) is an excluded minor for the class of frame matroidals.*

Our search for the remaining excluded minors of connectivity 2 for the class of frame matroids is therefore equivalent to the problem of finding excluded minors for the class of frame matroidals.

There are four ways to represent the 3-circuit $U_{2,3}$ as a biased graph: a balanced triangle, a contrabridged theta on two vertices, a tight handcuff consisting of an

unbalanced 2-cycle together with an unbalanced loop, or as a loose handcuff consisting of a link and two unbalanced loops; no biased graph representation of $U_{2,3}$ has all three elements as unbalanced loops. Evidently therefore, $U_{2,3} \oplus_2^{E(U_{2,3})} U_{2,4}$ is not frame. Let us denote this matroid N_9 . *I.e.*,

$$N_9 = U_{2,3} \oplus_2^{E(U_{2,3})} U_{2,4}.$$

Proposition 3.11. *N_9 is an excluded minor for the class of frame matroids.*

Proof. By Corollary 3.10, N_9 is an excluded minor for the class of frame matroids if and only if $(U_{2,3}, E(U_{2,3}))$ is an excluded minor for the class of frame matroids. There is no biased graph representing $U_{2,3}$ in which all three elements are unbalanced loops, so the matroidal $(U_{2,3}, E(U_{2,3}))$ is not frame. For every two element subset $L \subseteq E(U_{2,3})$ the matroidal $(U_{2,3}, L)$ is frame: a link between two vertices together with an unbalanced loop on each endpoint, where the two unbalanced loops represent the two elements in L is an L -biased graph representing $U_{2,3}$. \square

The matroid N_9 is the only excluded minor for the class of frame matroids of the form $N \oplus_2^L U_{2,4}$ with $|L| \geq 3$:

Theorem 3.12. *Let N be a 3-connected matroid, let $L \subseteq E(N)$, and suppose that $M = N \oplus_2^L U_{2,4}$ is an excluded minor for the class of frame matroids. If $|L| \geq 3$ then $M \cong N_9$.*

Proof. Let $L = \{e_1, \dots, e_k\}$. By Corollary 3.10, (N, L) is an excluded minor for the class of frame matroids. By minimality then, $(N, \{e_2, \dots, e_k\})$ is frame. Let Ω be a $\{e_2, \dots, e_k\}$ -biased graph representing $(N, \{e_2, \dots, e_k\})$. In Ω , edges e_2, e_3 are unbalanced loops and e_1 is a link. Since N is 3-connected, Ω is 2-connected. Hence there are disjoint paths P, Q linking the endpoints of e_1 and the vertices incident to e_2 and e_3 . Contracting P and Q yields $U_{2,3}$ as a minor with $E(U_{2,3}) = \{e_1, e_2, e_3\}$. By minimality and Proposition 3.11 therefore, $N \cong U_{2,3}$ and $L = \{e_1, e_2, e_3\}$. \square

4 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1.

Theorem 1.1. *Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. Then either M is isomorphic to a matroid in \mathcal{E} or M is the 2-sum of a 3-connected non-binary frame matroid and $U_{2,4}$.*

The set \mathcal{E} of excluded minors in the statement of Theorem 1.1 contains \mathcal{E}_0 and N_9 . In this section we exhibit the remaining matroids in \mathcal{E} , and show that any other excluded minor of connectivity 2 is a 2-sum of a 3-connected non-binary matroid and

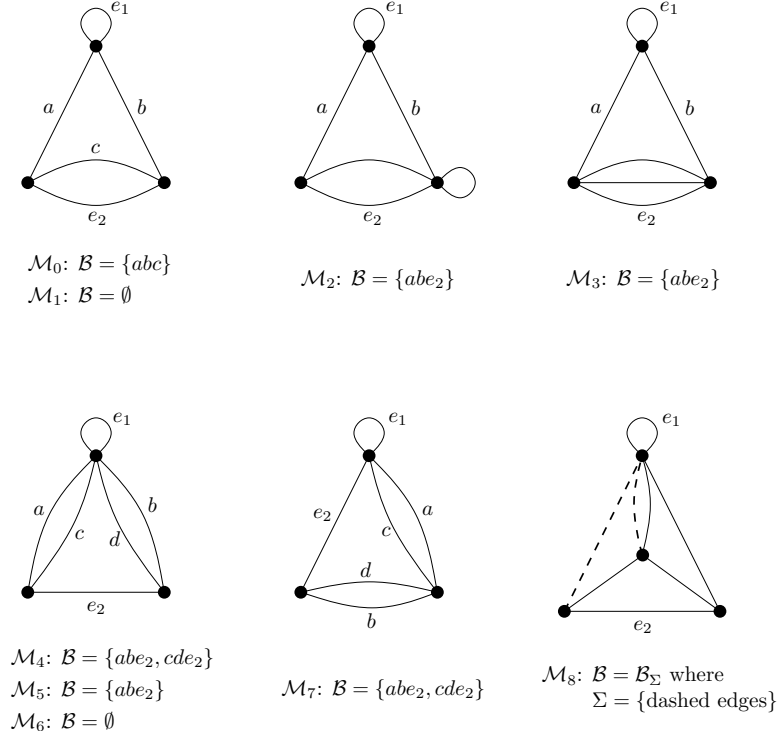


Figure 12: Excluded minors for the class of frame matroids with $|L| > 1$.

$U_{2,4}$. We do this using matroids. We show that the nine matroids $\mathcal{M}_0, \dots, \mathcal{M}_8$ illustrated in Figure 12 are excluded minors for the class of frame matroids. Each matroidal $\mathcal{M}_i = (M_i, L_i)$, $i \in \{0, \dots, 8\}$, is given as the frame matroid $M_i = F(\Omega_i)$ represented by a biased graph $\Omega_i = (G, \mathcal{B})$, where the graph G is shown in Figure 12 and collections \mathcal{B} are as listed. Each matroidal's set L_i is the set $\{e_1, e_2\}$, consisting of the pair of elements represented by edges e_1, e_2 in each graph.

Note that the excluded minor N_9 is given by the matroidal $\mathcal{M}_0 = (M_0, L_0)$: $M_0 \oplus_2^{L_0} U_{2,4} \cong N_9$ (it is straightforward to verify that the circuits of $M_0 \oplus_2^{L_0} U_{2,4}$ and those of N_9 coincide). In fact, $U_{2,3}$ gives rise to four excluded minors for the class of frame matroids, each yielding N_9 as corresponding excluded minor for the class of frame matroids, as follows. Write $E = E(U_{2,3})$, choose a subset $S \subseteq E$, and let $N = U_{2,3} \oplus_2^S U_{2,4}$. Set $L = E \setminus S$. Then $N \oplus_2^L U_{2,4} \cong N_9$ and matroidal (N, L) is an excluded minor for the class of frame matroids. The four choices for the size of S each thereby yield an excluded minor for the class of frame matroids.

Matroidals $\mathcal{M}_1, \dots, \mathcal{M}_8$ give rise to excluded minors for the class of frame matroids that we have not yet encountered. Let $\mathcal{E}_1 = \{N \oplus_2^L U_{2,4} \mid (N, L) \in$

$\{\mathcal{M}_0, \dots, \mathcal{M}_8\}$. Let $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$.

The hard work in proving Theorem 1.1 is in showing that $\{\mathcal{M}_1, \dots, \mathcal{M}_8\}$ is the complete list of excluded minors for the class of frame matroids having $|L| = 2$. This is the content of Lemma 4.1.

Lemma 4.1. *Let N be a 3-connected matroid and let $L \subseteq E(N)$ with $|L| = 2$. If (N, L) is an excluded minor for the class of frame matroids, then it is isomorphic to one of $\mathcal{M}_1, \dots, \mathcal{M}_8$.*

Before proving Lemma 4.1, let us show that it implies Theorem 1.1.

Proof of Theorem 1.1. Let M be an excluded minor for the class of frame matroids, and suppose M is not 3-connected. By Theorem 3.2, either M is isomorphic to a matroid in \mathcal{E}_0 or $M = N \oplus_2^L U_{2,4}$ for a 3-connected frame matroid N and a nonempty set L . So suppose $M \notin \mathcal{E}_0$. By Theorem 3.12, if $|L| \geq 3$ then $M \cong N_9$. If $|L| = 1$ then M is a 2-sum of N and $U_{2,4}$, and by Lemma 3.4 N is a non-binary. Finally, if $|L| = 2$, then by Corollary 3.10 (N, L) is an excluded minor for the class of frame matroids. By Lemma 4.1, M is isomorphic to a matroid in \mathcal{E}_1 . \square

4.1 The excluded minors \mathcal{E}_1

Let us substantiate our claim that the matroids in \mathcal{E}_1 are excluded minors for the class of frame matroids.

Say a matroid M *series reduces* to a matroid M' if M' may be obtained from M by repeatedly contracting elements contained in a nontrivial series class. Series reduction of matroids is useful because matroids consisting of a rank 2 matroid with a distinguished subset L of size 2 are always frame:

Lemma 4.2. *Let (N, L) be a matroidal. If N has rank 2 and $|L| = 2$, then (N, L) is frame.*

Proof. We may assume N has no loops. Let $L = \{e_1, e_2\}$. Since N has rank 2, N is obtained from some uniform matroid $U_{2,m}$ by adding elements in parallel. We may assume that either $e_1, e_2 \in E(U_{2,m})$ or that $e \in E(U_{2,m})$ and e_1 and e_2 are in the same parallel class. Let Ω be the contrabalanced biased graph representing $U_{2,m}$ with $V(\Omega) = \{u, v\}$, e_1 a loop incident to u , e_2 a loop incident to v if $e_2 \in E(U_{2,3})$, and all other elements represented by u - v edges. Let Ω' be the biased graph obtained by adding each element $f \in E(N) \setminus E(U_{2,m})$ in the same parallel class as an element $e \neq e_1$ as a u - v edge and declaring circuit ef balanced, and adding each element in $E(N) \setminus E(U_{2,3})$ in the same parallel class as e_1 as an unbalanced loop incident to u . Then Ω' is an L -biased representation of N . \square

This tool in hand, we may now prove:

Proposition 4.3. *The matroids $\mathcal{M}_0, \dots, \mathcal{M}_7$ are excluded minors for the class of frame matroids.*

Proof. That \mathcal{M}_0 is an excluded minor follows immediately from Corollary 3.10, Proposition 3.11, and the fact that $M_0 \oplus_2^{L_0} U_{2,4} \cong N_9$. So suppose for a contradiction that for some $i \in \{1, \dots, 8\}$, Ω is a L_i -biased graph representing $\mathcal{M}_i = (M_i, L_i) \in \{\mathcal{M}_1, \dots, \mathcal{M}_7\}$. Let e_1, e_2 be the elements in L_i . For $j \in \{1, 2\}$, let v_j be the vertex of Ω incident to e_j . Since $\{e_1, e_2\}$ is not a circuit, $v_1 \neq v_2$. Since each of M_1, \dots, M_7 has rank 3, $|V(\Omega)| = 3$; let u be the third vertex of Ω . Since none of M_1, \dots, M_7 has a circuit of size three containing e_1 and e_2 , there cannot be an edge linking v_1 and v_2 . But then u is a cut-vertex of Ω , a contradiction since all of M_1, \dots, M_7 are 3-connected.

We now show that every proper minor of each of $\mathcal{M}_1, \dots, \mathcal{M}_7$ is frame. The biased graphs shown in Figure 12 show that each matroidal $(M_i, L_i \setminus e_2)$ is frame ($i \in \{1, \dots, 7\}$). The biased graphs shown in Figure 13 show that also each matroidal $(M_i, L_i \setminus e_1)$ is frame. Any matroidal (N, L) obtained from one of $\mathcal{M}_1, \dots, \mathcal{M}_7$ by contracting an element other than e_1 or e_2 has matroid N of rank 2, and so is frame by Lemma 4.2. Finally, suppose that (N, L) is a matroidal obtained from one of $\mathcal{M}_1, \dots, \mathcal{M}_7$ by deleting an element e other than e_1, e_2 . In all cases, the resulting matroid series reduces to a matroid N' of rank 2 with both $e_1, e_2 \in E(N')$ by the contraction of a single edge s (this is easy to see by considering the biased graph representations of Figure 13: in each case, one of the biased graphs representing $M_i \setminus e$ obtained by deleting an edge $e \in \{a, b, c, d\}$ has a vertex incident to just two edges). By Lemma 4.2 therefore, there is an L -biased representation Ω' of the series reduced matroid N' . Now let Ω be a biased graph obtained from Ω' by placing an edge representing s in series with the other edge t in its series class in $M_i \setminus e$ — that is, if t is a link, subdivide t to produce a path consisting of edges s and t , and if t is a loop, say incident to v , add a vertex w , add s as a v - w link, and place t as a loop incident to w . Evidently this corresponds to a coextension of N' to recover N , and Ω is an L -biased representation of N . \square

Proposition 4.4. *The matroidal \mathcal{M}_8 is an excluded minor for the class of frame matroidals.*

Proof. The matroid of \mathcal{M}_8 is the rank 4 wheel, *i.e.* the cycle matroid $M(W_4)$ where W_4 is the five vertex simple graph having one vertex of degree 4 and four vertices of degree 3 (Figure 14). Its distinguished subset $L = \{e_1, e_2\}$ consists of two non-adjacent edges both of which have both ends of degree three. (Pinch the two ends of e_1 to obtain the biased graph representation shown in Figure 12.) We first show that \mathcal{M}_8 is not frame. Suppose for a contradiction that $M(W_4) \cong F(\Omega)$ for some L -biased graph Ω . Then e_1 and e_2 are both unbalanced loops in Ω ; say e_i is incident to vertex u_i ($i \in \{1, 2\}$). There is a unique circuit C of size 4 in $M(W_4)$ containing $\{e_1, e_2\}$; say $C = e_1 e_2 f f'$. Elements f, f' must form a path of length 2 in Ω linking u_1 and u_2 , say with interior vertex v . Since Ω is not balanced, $|V(\Omega)| = 4$; let v' be the fourth vertex of Ω . Note that since e_1, e_2 are not in a circuit of size 3 and not in any other circuit of size 4, all four remaining edges (other than e_1, e_2, f, f') must

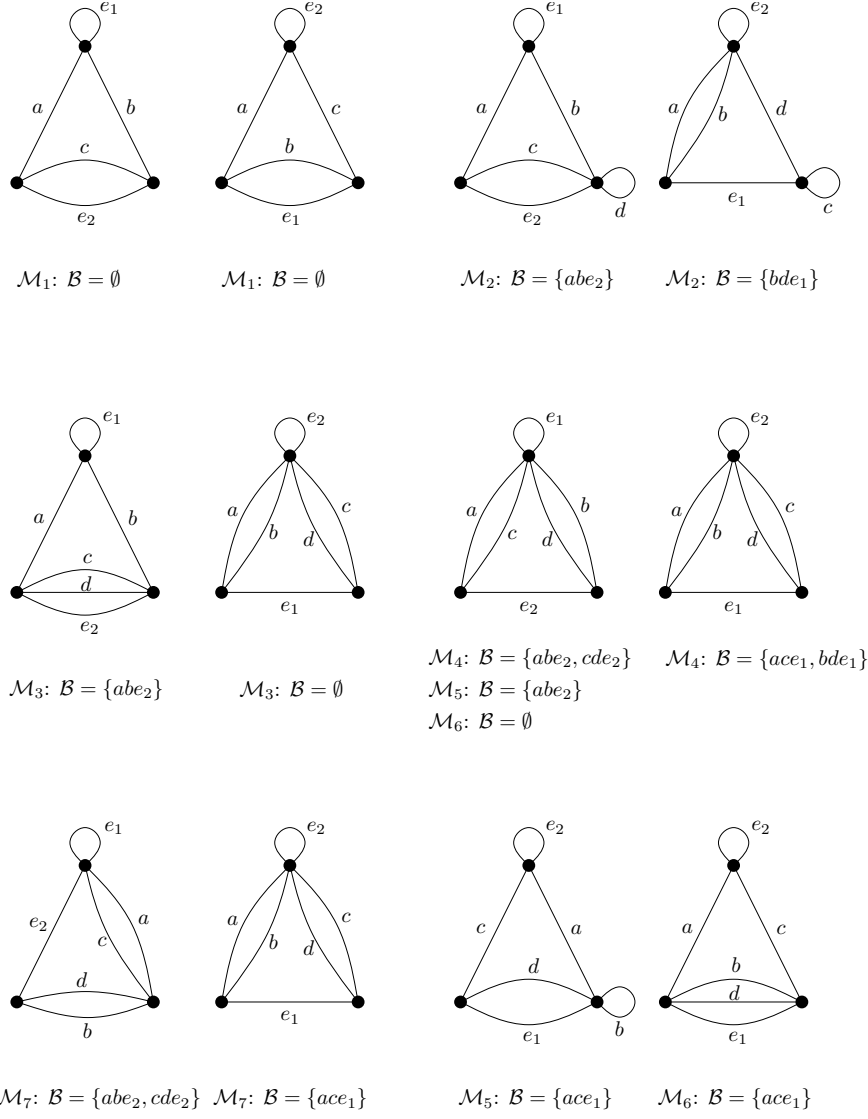


Figure 13: Alternate representations of $\mathcal{M}_1, \dots, \mathcal{M}_7$.

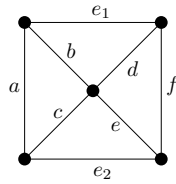


Figure 14: W_4 .

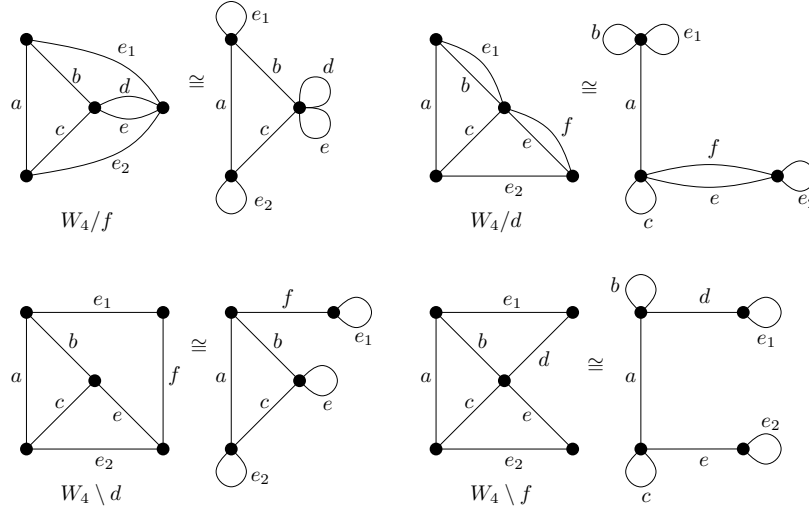


Figure 15: Any proper minor of W_4 is $\{e_1, e_2\}$ -biased.

be incident to v' . Since $M(W_4)$ has no elements in series or in parallel, there must be an edge with ends u_1, v' and another edge with ends u_2, v' . This yields another 4-circuit in $F(\Omega)$ containing e_1 and e_2 , a contradiction.

We now show that every proper minor of \mathcal{M}_8 is frame. For $i \in \{1, 2\}$, an $(L \setminus e_i)$ -biased graph is obtained by pinching the ends of e_{3-i} in the graph W_4 , so the matroidal $(M(W_4), L \setminus e_i)$ is frame. Now consider a matroidal obtained from \mathcal{M}_8 by deleting or contracting an element not in L . Up to symmetry there are only two such edges to consider, say elements d and f as shown in Figure 14. The biased graphs of Figure 15 show that deleting or contracting either of d or f yields a frame matroidal. These L -biased graphs may be obtained as follows.

- Contracting f in W_4 yields a graph in which e_1 and e_2 are incident to a common vertex. Rolling up the edges incident to that vertex yields an $\{e_1, e_2\}$ -biased graph, so $(M(W_4)/f, \{e_1, e_2\})$ is frame.
- In $M(W_4)/d$ elements $\{e, f\}$ are parallel. In $M(W_4)/d \setminus f$, elements e and e_2 are in series, so $M(W_4)/d$ is represented by the graph obtained from W_4/d by replacing e_2 with the pair of parallel edges e, f and replacing the pair e, f with e_2 . This yields a graph in which e_1 and e_2 are incident to a common vertex v . Now rolling up the edges in $\delta(v)$ yields an $\{e_1, e_2\}$ -biased graph representing $M(W_4)/d$, so $(M(W_5)/d, \{e_1, e_2\})$ is frame.
- In $M(W_4) \setminus d$ elements e_1 and f are in series, so the biased graph Ω obtained from $W_4 \setminus d$ by swapping edges e_1 and f represents $M(W_4) \setminus d$. Since e_1 and e_2 are incident to a common vertex v in Ω , the biased graph obtained by rolling up the edges in $\delta(v)$ is an $\{e_1, e_2\}$ -biased graph representing $M(W_4) \setminus d$.

- Similarly, $M(W_4) \setminus f$ has series classes $\{e_1, d\}$ and $\{e_2, e\}$. Hence swapping edges e_1 and d , and swapping edges e_2 and e , we obtain a biased graph representing $M(W_4) \setminus f$ in which e_1 and e_2 are incident to a common vertex. Rolling up the edges e_1, e_2, b, c incident to that vertex yields an $\{e_1, e_2\}$ -biased graph representing $M(W_4) \setminus f$. \square

4.2 Finding matroidal minors using configurations

To prove Lemma 4.1, we suppose $\mathcal{N}=(N, L)$ is an excluded minor for the class of frame matroids with $|L| = 2$ that is not one of $\mathcal{M}_1, \dots, \mathcal{M}_8$. We then work with a biased graph Ψ representing N to derive the contradiction that (N, L) contains one of $\mathcal{M}_0, \dots, \mathcal{M}_8$ as a minor. When doing so, we are looking for biased graphs representing one of $\mathcal{M}_0, \dots, \mathcal{M}_8$. Some of $\mathcal{M}_0, \dots, \mathcal{M}_8$ share the same underlying graphs or have an underlying graph contained in the underlying graph of another (Figures 12 and 13). Since which of $\mathcal{M}_0, \dots, \mathcal{M}_8$ we find as a minor of \mathcal{N} is irrelevant, it is enough to determine the underlying graph of a minor of Ψ along with just enough information about the biases of its cycles to see that Ψ must contain one of $\mathcal{M}_0, \dots, \mathcal{M}_8$ as a minor. We formalize this as follows.

A *configuration* \mathcal{C} consists of a graph G with two distinguished edges e_1, e_2 , together with a set \mathcal{U} of cycles of G , which we call *unbalanced*. The configurations we find are those named $\mathcal{C}_1, \dots, \mathcal{C}_4, \mathcal{C}'_4, \mathcal{C}''_4, \mathcal{C}_5, \dots, \mathcal{C}_8$ in Figure 16, and $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}'_2$, and \mathcal{D}_3 in Figure 17. We say that a biased graph $\Omega=(G, \mathcal{B})$ *realises* configuration $\mathcal{C}=(G, \mathcal{U})$ if $\mathcal{B} \cap \mathcal{U} = \emptyset$. The following two lemmas guarantee that finding one of these configurations in Ψ implies that \mathcal{N} contains one of $\mathcal{M}_0, \dots, \mathcal{M}_8$ as a minor.

Lemma 4.5. *Let Ω be a biased graph that realises one of the configurations $\mathcal{C}_1, \dots, \mathcal{C}_4, \mathcal{C}'_4, \mathcal{C}''_4, \mathcal{C}_5, \dots, \mathcal{C}_8$. Then $(F(\Omega), \{e_1, e_2\})$ contains one of $\mathcal{M}_0, \dots, \mathcal{M}_8$ as a minor.*

Proof. We show that in each case, Ω has a minor containing $\{e_1, e_2\}$ isomorphic to one of the biased graphs Ω_i representing the matroid M_i of a matroidal \mathcal{M}_i ($i \in \{0, \dots, 8\}$). This implies that $F(\Omega)$ has M_i as a minor containing $\{e_1, e_2\}$, and so that $(F(\Omega), \{e_1, e_2\})$ contains \mathcal{M}_i as a minor. Recall that the biased graphs Ω_i defining M_i ($i \in \{0, \dots, 8\}$) are those shown in Figure 12.

The only two realisations of \mathcal{C}_1 are the biased graphs Ω_0 and Ω_1 representing the matroids M_0 of \mathcal{M}_0 and M_1 of \mathcal{M}_1 . A biased graph realising \mathcal{C}_2 (resp. \mathcal{C}_3) will have a subgraph realising \mathcal{C}_1 unless it is isomorphic to Ω_2 (resp. Ω_3). A biased graph realising \mathcal{C}_4 has either 0, 1, or 2 balanced cycles, and so is isomorphic to one of Ω_4, Ω_5 , or Ω_6 , respectively. If Ω is a biased graph realising \mathcal{C}'_4 or \mathcal{C}''_4 then Ω has a unique balancing vertex after deleting its unbalanced loops; unrolling its unbalanced loops we obtain a biased graph Φ realising \mathcal{C}_4 with $F(\Phi) \cong F(\Omega)$.

Suppose Ω realises \mathcal{C}_5 . Let a, b be the two parallel edges forming the unbalanced cycle. We may assume by possibly interchanging a and b that the unique triangle containing a is unbalanced. Contracting a and deleting b yields a \mathcal{C}'_4 configuration.

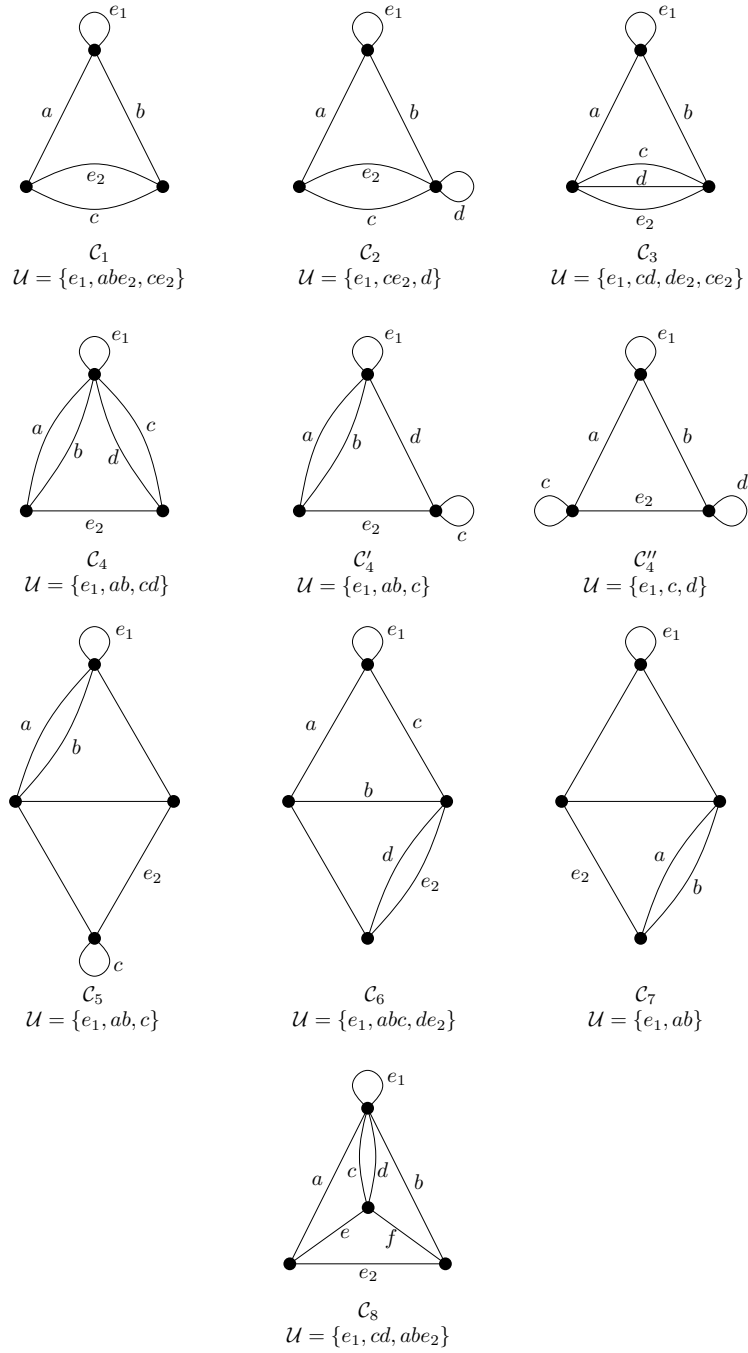


Figure 16: Configurations used to find $\mathcal{M}_0, \dots, \mathcal{M}_8$.

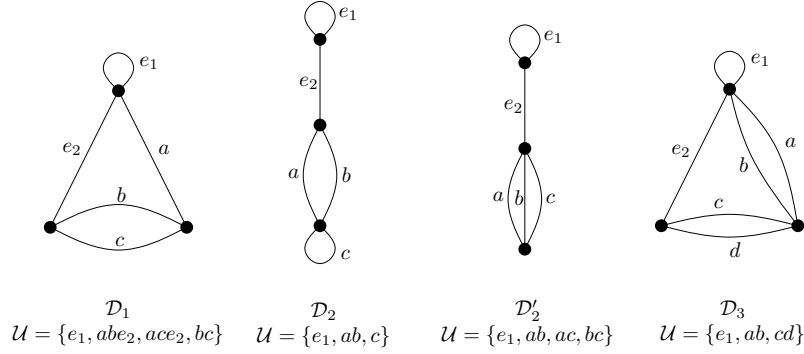


Figure 17: More configurations.

Suppose Ω realises \mathcal{C}_6 . Then by the theta property there is an unbalanced cycle either of length 3 or length 4 containing e_2 . In either case, this unbalanced cycle together with unbalanced cycle de_2 has a minor that is a \mathcal{C}_1 configuration.

If Ω realises \mathcal{C}_7 , then — since by the theta property one of a or b is in an unbalanced triangle — contracting one of edges a or b we obtain a \mathcal{C}_2 configuration.

Finally suppose that Ω realises \mathcal{C}_8 . If the triangle efe_2 is unbalanced, then deleting c, d and contracting one of the edges now in series yields configuration \mathcal{C}_1 . So suppose triangle efe_2 is balanced. If one of c or d — say d — fails to be contained in a balanced triangle, then deleting c and contracting d yields configuration \mathcal{C}_4 . The remaining possibility is that efe_2 is balanced and both c and d are contained in a balanced triangle. Then Ω may be embedded in the plane as drawn in Figure 16 with precisely facial cycles efe_2 , ace , and bdf balanced. The theta property implies that every cycle of length > 1 in this graph is unbalanced if in the embedding its interior contains the face bounded by unbalanced cycle cd , and is otherwise balanced. Hence $\Omega \cong \Omega_8$. \square

Lemma 4.6. *Let Ω be a biased graph which realises one of the configurations \mathcal{D}_1 , \mathcal{D}_2 , \mathcal{D}_2' , or \mathcal{D}_3 . Then $(F(\Omega), \{e_1, e_2\})$ contains one of \mathcal{M}_0 , \mathcal{M}_1 , or \mathcal{M}_7 as a minor.*

Proof. If Ω realises \mathcal{D}_1 then $F(\Omega) \cong M_1$, so $(F(\Omega), \{e_1, e_2\})$ is isomorphic to \mathcal{M}_1 . If Ω realises either \mathcal{D}_2 or \mathcal{D}_2' then $F(\Omega) \cong M_0$, so $(F(\Omega), \{e_1, e_2\})$ is isomorphic to \mathcal{M}_0 . If Ω realises \mathcal{D}_3 , then either Ω contains a \mathcal{D}_1 configuration or $F(\Omega) \cong M_7$ so $(F(\Omega), \{e_1, e_2\})$ is isomorphic to \mathcal{M}_7 . \square

Two of the excluded minors for the class of frame matroids have graphic matroids, namely \mathcal{M}_4 and \mathcal{M}_8 : M_4 is the cycle matroid of K_4 and M_8 is the rank 4 wheel. The following lemma will help us locate either \mathcal{M}_4 or \mathcal{M}_8 as a minor in a purported excluded minor $\mathcal{N}=(N, L)$ in which N is graphic.

Lemma 4.7. *Let G be a simple 3-connected graph, and let $\{e_1, e_2\} \subseteq E(G)$, with $e_1 = s_1 t_1$ and $e_2 = s_2 t_2$, with s_1, s_2, t_1, t_2 pairwise distinct. Then either G has a K_4 minor containing $\{e_1, e_2\}$ in which e_1 and e_2 do not share an endpoint, or G has W_4 as a minor containing $\{e_1, e_2\}$ in which e_1 and e_2 are opposite each other in the rim of W_4 (i.e., e_1 and e_2 do not share an endpoint and each of e_1 and e_2 have both endpoints of degree three).*

Proof. Let $\text{co}(H)$ denote the graph obtained from a graph H by suppressing vertices of degree 2. It is well known that if G is a 3-connected graph, then for every $e \in E(G)$, either $\text{co}(G \setminus e)$ or G/e is 3-connected (for instance, it is a special case of Proposition 8.4.6 in [8]). In the following, if in $G \setminus e$ edge e_i , $i \in \{1, 2\}$, has an endpoint of degree two, then $\text{co}(G \setminus e)$ is obtained by contracting the edge other than e_i incident to that vertex.

Let G be a minimal counter-example to the statement of the lemma. If there is an edge $e \in E(G)$ such that $\text{co}(G \setminus e)$ or G/e is 3-connected such that e_1 and e_2 are not incident to a common vertex, then by minimality this graph has a minor of one of the required forms. But then so would G have had that minor, a contradiction. Hence for every edge $e \notin \{e_1, e_2\}$, if $\text{co}(G \setminus e)$ is 3-connected then e_1 and e_2 are adjacent in $\text{co}(G \setminus e)$, and if G/e is 3-connected then e_1 and e_2 are adjacent in G/e .

Suppose there is an edge $e \in E(G)$ that does not have any of s_1, t_1, s_2, t_2 as an endpoint. Then $\text{co}(G \setminus e)$ has e_1 and e_2 nonadjacent, and so is not 3-connected. Hence G/e is 3-connected. But neither are e_1 and e_2 adjacent in G/e , contradicting the previous paragraph. Therefore every edge of G has an endpoint incident to e_1 or e_2 . Now suppose $e \in E(G)$ does not have both endpoints in $\{s_1, t_1, s_2, t_2\}$; say $e = x s_1$ with $x \notin \{s_1, t_1, s_2, t_2\}$. Then G/e does not have e_1 and e_2 adjacent, and so is not 3-connected. Hence $\text{co}(G \setminus e)$ is 3-connected, and so has e_1 and e_2 adjacent. This implies that the degree of s_1 is three, and the three edges incident to s_1 are e , e_1 , and f , where the other endpoint of f is one of s_2 or t_2 . It follows that $|V(G)| \leq 5$. (Every vertex $x \notin \{s_1, t_1, s_2, t_2\}$ has neighbourhood of size ≥ 3 contained in $\{s_1, t_1, s_2, t_2\}$. Further, each vertex in the neighbourhood of x has degree three, which, together with its edge to x and its incident edge in $\{e_1, e_2\}$, includes an edge whose other endpoint is also in $\{s_1, t_1, s_2, t_2\}$. These edges resulting from the existence of $x \notin \{s_1, t_1, s_2, t_2\}$ accounted for thus far leave just one vertex u in $\{s_1, t_1, s_2, t_2\}$ for which it is possible that u has an additional incident edge, yet the existence of a vertex $y \notin \{x, s_1, t_1, s_2, t_2\}$ requires three such vertices.)

If $|V(G)| = 4$, then $G \cong K_4$ and we are done. So suppose $|V(G)| = 5$; let $V(G) = \{x, s_1, t_1, s_2, t_2\}$. The fact that the degree of every vertex is at least three, together with the above constraints on edges incident to a neighbour of x forces the existence of either a K_4 or W_4 minor of the required form. This contradiction completes the proof. \square

4.3 Proof of Lemma 4.1

If a biased graph Ω has a minor realising a configuration, we say Ω *contains* the configuration. Let us call the configurations $\mathcal{C}_1, \dots, \mathcal{C}_4, \mathcal{C}'_4, \mathcal{C}''_4, \mathcal{C}_5, \dots, \mathcal{C}_8, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}'_2, \mathcal{D}_3$ *bad* configurations. Thus by Lemmas 4.5 and 4.6, if Ω represents M , and Ω contains a bad configuration, then the matroidal $(M, \{e_1, e_2\})$ has one of $\mathcal{M}_0, \dots, \mathcal{M}_8$ as a minor.

Proof of Lemma 4.1. Let $\mathcal{N}=(N, L)$ be an excluded minor for the class of frame matroidals with N 3-connected and $L=\{e_1, e_2\}$, and suppose \mathcal{N} is not isomorphic to one of $\mathcal{M}_1, \dots, \mathcal{M}_8$. Observe that \mathcal{N} cannot have \mathcal{M}_0 as a minor, since then minimality would imply that $\mathcal{N} \cong \mathcal{M}_0$; since e_1 and e_2 are in series in M_0 this would contradict the fact that N is 3-connected. In light of this and Lemmas 4.5 and 4.6 it suffices to derive the contradiction that a biased graph Ω representing N contains a bad configuration.

First suppose that N is graphic. Let H be a graph with $N = M(H)$. As N is 3-connected, H is simple and 3-connected. Hence neither e_1 nor e_2 is a loop in H . If edges e_1 and e_2 share an endpoint $v \in V(H)$, then rolling up the edges incident to v yields a biased graph in which both e_1 and e_2 are unbalanced loops, a contradiction. Hence e_1 and e_2 do not share an endpoint. By Lemma 4.7 therefore, H has a minor H' isomorphic to either K_4 with e_1 and e_2 nonadjacent, or isomorphic to W_4 with e_1 and e_2 nonadjacent and neither incident to the vertex of degree 4. In the former case \mathcal{N} contains \mathcal{M}_4 as a minor, and in the latter \mathcal{M}_8 as a minor, both contradictions.

So N is not graphic. Let $\Omega=(G, \mathcal{B})$ be a biased graph representing $(N, \{e_1\})$. Since N is 3-connected:

- (C1) Ω is 2-connected, and
- (C2) if (A, B) is a separation of N with $|A| \geq 2$ and $\Omega[A]$ is balanced, then $|V(A) \cap V(B)| \geq 3$.

Let v be the vertex to which e_1 is incident. We consider two cases, depending on whether e_1 and e_2 are adjacent in Ω .

4.3.1 Case 1. e_1 and e_2 are not adjacent

Let u, w be the endpoints of e_2 . We consider three subcases depending on the behaviour of unbalanced cycles in $\Omega - v$.

Subcase (i) $\Omega - v$ has no unbalanced cycle of length > 1

If $\Omega - v$ contains unbalanced loops, then unrolling them yields an $\{e_1\}$ -biased graph representing N in which v is a balancing vertex. We may assume therefore that $\Omega - v$ is balanced. Consider the balancing equivalence classes in $\delta(v)$. There cannot be just one b-class in $\delta(v)$, since then e_1 would not be contained in any circuit of N . If there are only two b-classes, then by Proposition 1.6 Ω is a signed

graph. But then splitting v yields a graph H with $M(H) = N$ (Proposition 1.10), so N is graphic, a contradiction. Hence there are at least three b-classes in $\delta(v)$.

Claim. Ω contains a \mathcal{C}_4 configuration.

Proof of claim. Construct an auxiliary graph G from the underlying graph of $\Omega - e_1$, as follows. Let $\{S_1, \dots, S_t\}$ be the partition of $\delta(v)$ into its b-classes. Add a set of new vertices $X = \{x_1, \dots, x_t\}$, and, for each $i \in \{1, \dots, t\}$, redefine the endpoints of each edge $f = xv \in S_i$ so that f has endpoints x, x_i . Add a new vertex y to G that is adjacent to every vertex which is a neighbour of either u or w .

We claim that G contains three vertex disjoint paths between X and $\{u, w, y\}$. For if not, then by Menger's Theorem there exists a pair of subgraphs $G_1, G_2 \subseteq G$ whose edges partition $E(G)$ so that $X \subseteq V(G_1)$ and $\{u, w, y\} \subseteq V(G_2)$ and $V(G_1) \cap V(G_2) = Z$ with $|Z| = 2$. If Z contains at most one vertex of X then the subgraph of Ω induced by $E(G_2)$ is balanced, contradicting (C2). Hence Z contains two vertices of X . But this implies v is a cut vertex of Ω , contradicting (C1). This establishes the existence of our paths.

So we may now assume that in Ω there exist three internally disjoint paths, P_1 and P_2 from v to u and P_3 from v to w such that the three edges of these paths in $\delta(v)$ are in distinct b-classes. If there exists a path Q from $P_1 \cup P_2$ to P_3 which is disjoint from $\{u, v\}$, then a minor of $P_1 \cup P_2 \cup P_3 \cup Q \cup \{e_1, e_2\}$ contains a \mathcal{C}_4 configuration.

If there is no such path Q , then there is a partition (A, B) of $E(\Omega)$ with $V(A) \cap V(B) = \{u, v\}$, $P_1, P_2 \subseteq \Omega[A]$ and $P_3 \subseteq \Omega[B]$. Choose such a partition with B minimal. By (C2), $\Omega[B]$ contains two edges from $\delta(v)$ in distinct equivalence classes, and by our choice of B , neither of these edges is incident with u . Also by our choice of B , the subgraph $\Omega[B] - \{u, v\}$ is connected. It follows that Ω contains a \mathcal{C}_4 configuration. \square

This completes the proof in subcase 1(i).

Subcase (ii) $\Omega - v$ has an unbalanced cycle of length > 1 but none containing e_2

Since two vertex disjoint paths linking the endpoints of e_2 and an unbalanced cycle would, by the theta property, yield an unbalanced cycle containing e_2 , in this case $\Omega - v$ is not 2-connected. We investigate the block structure of $\Omega - v$ to show that Ω contains a bad configuration.

Suppose Ψ is a leaf block of $\Omega - v$, containing cut-vertex x . By (C1) there is at least one edge between v and $\Psi - x$. By (C2), either Ψ is unbalanced or there exists an unbalanced cycle C containing v with length > 1 with $C - v \subseteq \Psi$. With the goal of finding a bad configuration in mind, edges of Ψ may be deleted or contracted to yield, in the former case, an unbalanced loop at x and a link vx , or in the latter case, two vx links forming an unbalanced cycle.

Let Φ be the block of $\Omega - v$ containing e_2 . Suppose first that Φ is not a leaf block of $\Omega - v$. Then Φ contains two distinct cut-vertices x, x' . Choose a path in this block linking x and x' and containing e_2 . Applying the argument of the previous paragraph to two leaf blocks of $\Omega - v$, we find that Ω contains one of the configurations \mathcal{C}_4 , \mathcal{C}'_4 , or \mathcal{C}''_4 .

So suppose now that the block Φ of $\Omega - v$ is a leaf block. After deleting unbalanced loops Φ is balanced, else Φ (and so $\Omega - v$) would contain an unbalanced cycle containing e_2 . Let x be the cut vertex of $\Omega - v$ contained in Φ , and let S be the set of edges in $\delta(v)$ incident with a vertex of $\Phi - x$. Consider the biased graph Φ' obtained from Φ by deleting its unbalanced loops and adding vertex v together with the edges in S . Vertex v is a balancing vertex of Φ' ; let $\{S_1, \dots, S_t\}$ be the partition of S into the b-classes of $\delta(v)$ in Φ' . Let S_0 be the set of loops in Φ not incident to x .

Now construct an auxiliary graph similar to that appearing in subcase 1(i). Let G be the graph obtained from Φ by adding vertices x_0, x_1, \dots, x_t , and for $1 \leq i \leq t$ and every edge $zv \in S_i$ add an edge zx_i ; for each unbalanced loop incident to a vertex z add an edge zx_0 . Finally, add a vertex y that is adjacent to each vertex which is a neighbour of either u or w . We claim that in G there exist three vertex disjoint paths linking $\{x, x_0, \dots, x_t\}$ to $\{u, w, y\}$. For suppose otherwise. Then by Menger's Theorem there exists a pair of subgraphs $G_1, G_2 \subseteq G$ whose edges partition $E(G)$ with $\{x, x_0, \dots, x_t\} \subseteq V(G_1)$ and $\{u, w, y\} \subseteq V(G_2)$ and $|V(G_1) \cap V(G_2)| = 2$. Let $Z = V(G_1) \cap V(G_2)$. If both vertices in Z are in $\{x_0, x_1, \dots, x_t\}$, then $\Omega - v$ would have no path linking x and u , contradicting the fact that Φ is a block of $\Omega - v$. Now either $x_0 \notin Z$ or $x_0 \in Z$. If $x_0 \notin Z$, then in Ω the biased subgraph induced by $E(G_2 - y)$ is a balanced subgraph meeting the rest of Ω in just two vertices, contradicting (C2). But if $x_0 \in Z$, then the biased subgraph induced by $E(G_2 - y)$ meets the rest of Ω in just one vertex, contradicting (C1). Hence the paths exist as claimed.

We may assume that one of these three paths begins at vertex x (otherwise choose a path from x to $\{u, w, y\}$ modify a path appropriately). In Ω this gives us three internally disjoint paths $P_1, P_2, P_3 \subseteq \Phi'$ such that:

1. P_1, P_2 start at v or at a vertex of Φ incident with an unbalanced loop and end at $\{u, w\}$.
2. at least one of P_1, P_2 starts at v , and if both start at v their first edges are in distinct blancing classes.
3. P_3 starts at x and ends at $\{u, w\}$.
4. at least one of P_1, P_2, P_3 ends at u and one at w .

Choose an unbalanced cycle C of length > 1 in $\Omega - v$ and choose two vertex disjoint paths R, R' linking C and $\{v, x\}$. Note that C is not contained in Φ (as Φ without

its unbalanced loops is balanced), and so R, R' meet Φ only at x . First suppose that both P_1 and P_2 end at u or both end at w . Consider the subgraph H consisting of $C \cup R \cup R'$ together with $P_1 \cup P_2 \cup P_3$ and the edges e_1, e_2 . If both P_1, P_2 begin at v then H contains a \mathcal{C}_4 configuration. Otherwise, one of these paths begins at a vertex incident with an unbalanced loop f . Adding f to subgraph H , we find that H contains \mathcal{C}'_4 configuration. So now suppose that P_1 ends at u while both P_2 and P_3 end at w . Since Φ is a block of $\Omega - v$, $\Phi - w$ contains a path Q from $P_1 - v$ to $P_3 - \{v, w\}$. If Q contains a vertex in P_2 , then again the subgraph H consisting of $C \cup R \cup R'$ together with $P_1 \cup P_2 \cup P_3 \cup \{e_1, e_2\}$ and possibly an unbalanced loop incident to an end of P_1 or P_2 , contains either a \mathcal{C}_4 or \mathcal{C}'_4 configuration. Otherwise, H contains either configuration \mathcal{C}_5 (if one of P_1 or P_2 does not begin at v but is incident to an unbalanced loop) or \mathcal{C}_8 (if both P_1 and P_2 begin at v).

Subcase (iii) $\Omega - v$ has an unbalanced cycle containing e_2

Let C be an unbalanced cycle containing e_2 . Choose two paths P_1, P_2 linking v and C , disjoint except at v , say meeting C at vertices x_1, x_2 , respectively. Let R be the x_1 - x_2 path in C containing e_2 ; let R' be the x_1 - x_2 path in C avoiding e_2 . If the cycle $P_1 \cup P_2 \cup R$ is unbalanced, then Ω contains configuration \mathcal{C}_1 . So let us now assume that this does not occur for any unbalanced cycle containing e_2 — *i.e.*, for every unbalanced cycle C of $\Omega - v$ containing e_2 and every such pair P_1, P_2 of v - C paths meeting only at v , the cycle formed by $P_1 \cup P_2$ and the path R in C traversing e_2 is balanced. Choose such subgraphs C, P_1 and P_2 , with P_1 meeting C at x_1 and P_2 meeting C at x_2 , so that the length of the path R' in C avoiding e_2 is minimum.

Suppose R' does not consist of a single edge. First suppose also that there exists a separation (Ω_1, Ω_2) of Ω with $V(\Omega_1) \cap V(\Omega_2) = \{x_1, x_2\}$ with $R' \subseteq \Omega_1$ and $P_1 \cup P_2 \cup R \subseteq \Omega_2$. By choosing such a separation with Ω_1 minimal, we may further assume that $\Omega_1 - \{x_1, x_2\}$ is connected and that there are no x_1x_2 edges in Ω_1 . By (C2), Ω_1 is not balanced. If there is an unbalanced cycle in $\Omega_1 - x_1$, then Ω contains a \mathcal{C}_2 configuration. Otherwise x_1 is a balancing vertex in Ω_1 . Since Ω_1 contains no x_1 - x_2 edge and $\Omega_1 - \{x_1, x_2\}$ is connected, there is then an unbalanced cycle in $\Omega_1 - x_2$; again we find a \mathcal{C}_2 configuration. So now assume that no such separation exists: there is a path Q from the interior of R' to $(P_1 \cup P_2 \cup R) \setminus \{x_1, x_2\}$. If Q first meets $P_1 \cup P_2 \setminus \{x_1, x_2\}$, then we find our choice of P_1 and P_2 did not minimise R' , a contradiction. Hence Q avoids $(P_1 \cup P_2) \setminus \{x_1, x_2\}$ and meets R . Subgraph $Q \cup C$ is a theta. If the cycle in $Q \cup C$ containing e_2 different from C is unbalanced, then again we did not choose C, P_1 , and P_2 so as to minimise the length of R' , a contradiction. Therefore that cycle is balanced, and so the cycle C' in $C \cup Q$ not containing e_2 is unbalanced. Choose an edge $e \in Q$. Contracting all edges of C' but e , all but one edge of $R' \setminus C'$, all but edge e_2 of $R \setminus C'$, and all but one edge of each of P_1 and P_2 , we find configuration \mathcal{C}_2 .

So the path R' must consist of a single x_1x_2 edge f . Suppose first that $\{x_1, x_2\}$ does not separate v from $C \setminus \{x_1, x_2\}$ and choose a path Q from $(P_1 \cup P_2) \setminus \{x_1, x_2\}$ to

$C \setminus \{x_1, x_2\}$. We claim that by the theta property, there exists a cycle in $P_1 \cup P_2 \cup Q \cup C$ containing both e_2 and Q which is unbalanced, and in any case this yields a \mathcal{C}_1 configuration. To see this, recall that the cycle $P_1 \cup P_2 \cup C \setminus f$ is balanced. There are, up to symmetry and assuming Q leaves from P_1 , two cases to consider: (a) Q is a P_1 - C path such that the cycle D in $P_1 \cup Q \cup C$ containing e_2 and Q contains f , or (b) does not contain f . In case (a), if D is balanced, then the cycle in $P_1 \cup P_2 \cup (C \setminus f) \cup Q$ containing Q and e_2 is unbalanced, and we find \mathcal{C}_1 contained in this cycle together with C and e_1 . If D is unbalanced, then we find \mathcal{C}_1 in $P_1 \cup Q \cup C \cup \{e_1\}$. In case (b), if D is balanced we find \mathcal{C}_1 by deleting the subpath of P_1 between $P_1 \cap Q$ and x_1 . If D is unbalanced, we find \mathcal{C}_1 in $P_1 \cup Q \cup C \cup \{e_1\}$.

Hence $\{x_1, x_2\}$ separates v from C . Choose a separation (Ω_1, Ω_2) of Ω with $V(\Omega_1) \cap V(\Omega_2) = \{x_1, x_2\}$ for which $C \subseteq \Omega_2$ and $v \in \Omega_1$, with Ω_1 minimal. Then $\Omega_1 - \{x_1, x_2\}$ is connected and Ω_1 has no x_1x_2 edge. If Ω_1 contains an unbalanced cycle C' of length > 1 , then choosing a pair of vertex disjoint paths Q, Q' linking C' and $\{x_1, x_2\}$ and an application of the theta property yield an unbalanced cycle containing e_2 that is not C . But then $C' \cup Q \cup Q' \cup C \cup \{e_1\}$ contains a \mathcal{C}_1 configuration. Hence Ω_1 contains no unbalanced cycle of length > 1 ; suppose Ω_1 contains an unbalanced loop $e \neq e_1$, say incident to v' . Since N is 3-connected, $v' \neq v$. Since Ω is 2-connected, there is a path Q from v' to $(P_1 \cup P_2) \setminus v$. But now in $C \cup P_1 \cup P_2 \cup Q \cup \{e\}$ we find configuration \mathcal{C}_2 .

So $\Omega_1 - e_1$ is balanced. Now suppose that $V(\Omega_2) = \{x_1, x_2\}$. If there is a loop in Ω_2 we have a \mathcal{C}_2 configuration. If there are at least three edges in Ω_2 we have a \mathcal{C}_3 configuration (no two such edges form a balanced cycle since N is 3-connected). So in this case Ω_2 consists only of the two edges e_2 and f (which form unbalanced cycle C). Since $\Omega_1 - e_1$ is balanced, an unbalanced cycle in $\Omega - f$ containing e_2 , together with the theta property, would yield a \mathcal{C}_1 configuration. Hence $\Omega - \{e_1, f\}$ is balanced. But this implies that e_1 and f are in series in N , a contradiction since N is 3-connected. So $|V(\Omega_2)| \geq 3$.

We now claim that Ω_2 contains an unbalanced cycle that does not contain both x_1 and x_2 . Let Ψ_0 be a component of $\Omega_2 - \{x_1, x_2\}$ and let Ψ be the subgraph of Ω_2 consisting of Ψ_0 together with all edges between x_i and $V(\Psi_0)$, for $i \in \{1, 2\}$. By (C2), Ψ is unbalanced. Moreover, we may assume x_1 is a balancing vertex in Ω_2 , since if not we have the desire cycle. Consider the b-classes of $\delta_{\Omega_2}(x_1)$. Since Ψ is not balanced, there are two edges in Ψ in distinct b-classes, and since $\Psi - x_2$ is connected, this yields an unbalanced cycle in Ω_2 not containing x_2 , as desired.

Without loss of generality, choose an unbalanced cycle $D \subseteq \Omega_2$ that does not contain x_1 . If D and C share at most one vertex, we see that $P_1 \cup P_2 \cup C \cup D$ contains a \mathcal{C}_2 configuration. So $|V(C) \cap V(D)| \geq 2$. Let Q be the maximal subpath of C which contains e_2 and has no interior vertex in the set $V(D) \cup \{x_1, x_2\}$. By assumption at least one end of Q must be in $V(D)$. If both ends of Q are in $V(D)$ then Ω_2 contains an unbalanced cycle D' containing e_2 but not x_1 . There are two vertex disjoint paths linking D' and $\{x_1, x_2\}$ and these, together with $P_1 \cup P_2 \cup \{e_1, f\}$,

contain a \mathcal{C}_6 configuration. So finally assume (by possibly interchanging x_1 and x_2) that one end of Q is x_1 and the other is in $V(D)$. The D - x_2 path in C avoids Q ; this path together with D , Q , f , P_1 , P_2 , and e_1 contains a \mathcal{C}_7 configuration.

This completes the proof of Case 1.

4.3.2 Case 2. e_1 and e_2 are adjacent

As before, let v be the endpoint of e_1 . Let u be the other endpoint of e_2 . Let T_0 be the standard block-cutpoint graph of $\Omega - v$. If u is a cut vertex of $\Omega - v$ then set $T = T_0$. Otherwise, let T be the tree obtained by adding vertex u to T_0 together with an edge between u and the unique block of $\Omega - v$ containing u . View tree T as rooted at u . Every block Ψ of $\Omega - v$ is a vertex of T and there is a unique path in T from Ψ to u . The next vertex of T on this path from Ψ is a vertex of Ω , the *parent* of Ψ . Note that the parent of a block of $\Omega - v$ is always either a cut vertex of $\Omega - v$ or is u .

Claim. If x is the parent of a block Ψ of $\Omega - v$, then one of the following holds:

1. Ψ contains no unbalanced cycle of length > 1 .
2. x is balancing in Ψ and there are exactly two b-classes in $\delta_\Psi(x)$.

Proof of Claim. Let Ψ' be the graph obtained from Ψ by deleting all loops. If Ψ' is balanced, (1) holds. Otherwise, suppose x is not a balancing vertex of Ψ' and choose an unbalanced cycle C of $\Psi' - x$ and two paths P_1, P_2 from x to C that are disjoint except at x . Let y_1, y_2 be the respective ends of P_1, P_2 on C , and let Q, Q' be the two paths in C meeting just at y_1 and y_2 . By the theta property, one of $P_1 \cup P_2 \cup Q$ or $P_1 \cup P_2 \cup Q'$ is unbalanced. Hence $P_1 \cup P_2 \cup C$ together with an x - u path, e_2 , and e_1 , contains a \mathcal{D}_2 configuration.

So x is balancing in Ψ' . If $\delta_\Psi(x)$ contains three b-classes, Ω contains a \mathcal{D}'_2 configuration. Hence there are exactly two b-classes in $\delta_\Psi(x)$. If Ψ contains an unbalanced loop not at vertex x , then an unbalanced cycle in Ψ' , together with this loop, an x - u path, e_2 , and e_1 , contains a \mathcal{D}_2 configuration. \square

Call a block of $\Omega - v$ as described in statement (1) of our claim a *type 1* block, and a block as in statement (2), a *type 2* block.

Claim. Every type 2 block of $\Omega - v$ is a leaf of T .

Proof of Claim. Suppose there exists a type 2 block Ψ of $\Omega - v$ that is not a leaf of T . Let Φ be a leaf block of $\Omega - v$ with parent y such that the unique path in T from Φ to u contains Ψ . If Φ contains an unbalanced cycle, then Ω contains a \mathcal{D}_2 configuration. So Φ is balanced. Let Φ^+ be the biased subgraph of Ω given by Φ together with v and all edges between v and $\Phi - y$. By (C2), Φ^+ is unbalanced, so there is an unbalanced cycle C in Φ^+ containing v . Together with a C - y path in Φ , an unbalanced cycle C' in Ψ , a y -($C' - x$) path and an x - u path in $\Omega - v$, we have a biased graph containing a \mathcal{D}_3 configuration. \square

Along with the structure we have determined of $\Omega - v$ comes knowledge of the biases of all cycles of $\Omega - v$. We wish to extend this knowledge to Ω .

Let Ω_0 be the balanced biased subgraph of Ω consisting of each type 1 block of $\Omega - v$ without its unbalanced loops. By our second claim, Ω_0 is a connected balanced biased subgraph of $\Omega - v$. Let Ψ_1, \dots, Ψ_m be the type 2 blocks of $\Omega - v$. For each Ψ_i , let x_i be its parent vertex in T , and define Ω_i to be the subgraph of Ω consisting of Ψ_i together with v and all edges between v and $\Psi_i - x_i$. Let U be the set of all loops in $\Omega - v$. The subgraphs $E(\Omega_0), E(\Omega_1), \dots, E(\Omega_m)$ are edge disjoint and together contain all edges in $E(\Omega)$ except for loops and some edges incident to v (see Figure 18, at left).

For every $1 \leq i \leq m$, vertex x_i is balancing in Ψ_i ; let $\{A_i, B_i\}$ be the partition of $\delta_{\Psi_i}(x_i)$ into its two b-classes. Suppose Ω_i contains an unbalanced cycle C disjoint from x_i . Choose two internally disjoint paths P_1, P_2 linking x_i and C for which $E(P_1) \cap A_i \neq \emptyset$ and $E(P_2) \cap B_i \neq \emptyset$. Then $C \cup P_1 \cup P_2 \cup \{e_1, e_2\}$ together with an x_i - u path in $\Omega - v$ contains a \mathcal{D}_3 configuration. Hence every Ω_i has x_i as a balancing vertex. By Lemma 1.5 the b-classes in each $\delta_{\Omega_i}(x_i)$ are $\{A_i, B_i\}$.

Consider two edges $f, f' \in A_i$ or $f, f' \in B_i$ for some $1 \leq i \leq m$. Let C (resp. C') be a cycle containing e_2 and f (resp. f'). The path $C - e_2$ ($C' - e_2$) is the union of a u - x_i path P (P') and an x_i - v path Q (Q'). Applying Lemma 1.5 separately to $P \cup P'$ and $Q \cup Q'$, we conclude that C and C' have the same bias. Now suppose that for some $1 \leq i \leq m$, there is an unbalanced cycle containing e_2 and an edge in A_i and another unbalanced cycle containing e_2 and an edge in B_i . Choose a cycle $C \subseteq \Psi_i$ that contains one edge in each of A_i and B_i , a path P in Ω_i from v to $C - x_i$, and a u - x_i path Q in Ω_0 . It now follows that $P \cup Q \cup C \cup \{e_1, e_2\}$ contains a \mathcal{D}_1 configuration. Hence two such unbalanced cycles do not exist, and by possibly interchanging the names assigned to the sets A_i, B_i , we may assume that for every $1 \leq i \leq m$, every cycle in Ω containing e_2 and an edge of A_i is balanced. By the theta property then, for every $1 \leq i < j \leq m$, every cycle of Ω containing an edge in A_i and an edge in A_j is balanced.

We now define a signature for Ω that realises \mathcal{B} . We use a simpler biased graph Ω' to model the biases of cycles in Ω to do so. Let Ω' be the biased graph obtained from Ω as follows. For every $1 \leq i \leq m$ replace Ω_i with two edges a_i, b_i with endpoints x_i and v , with $a_i \in A_i$ and $b_i \in B_i$, and let the bias of each cycle of Ω' be inherited from a corresponding cycle in Ω in the obvious way. Now $\Omega' - v$ has no unbalanced cycle of length > 1 , so by Observation 1.9, Ω' is a k -signed graph. Moreover, by Observation 1.9 there is a signature $\Sigma' = \{U, \Sigma'_1, \dots, \Sigma'_k\}$ that realises the biases of cycles of Ω' , where each set $\Sigma'_j \subseteq \delta_{\Omega'}(v)$ and U is the set of unbalanced loops of Ω' . Further, Observation 1.9 allows us to assume that e_2 is not a member of any set in the signature Σ' . Since for every $1 \leq i \leq m$, e_2 and a_i are in the same b-class, none of the edges a_i is in a member of the signature. Hence every b_i is contained in some member Σ'_j of Σ' . Define a signature $\Sigma = \{U, \Sigma_1, \dots, \Sigma_k\}$ for Ω as follows. For every $1 \leq i \leq m$, if $b_i \in \Sigma'_j$ put all edges in B_i in Σ_j . If $e = vz \in \Sigma'_j$ is

a edge incident to v and a vertex $z \in \Omega_0$, put e in Σ_j . The structural description we have of Ω and the biases of its cycles implies $\mathcal{B}_\Sigma = \mathcal{B}$. By Theorem 1.11, the biased graph Γ obtained by performing a twisted flip on Ω has $F(\Gamma) \cong F(\Omega)$ (Figure 18, at right). But in Γ both e_1 and e_2 are represented as unbalanced loops, so (N, L) is frame, a contradiction. \square

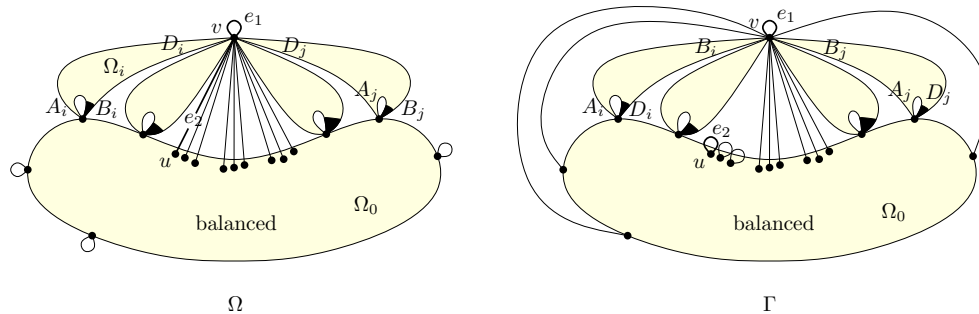


Figure 18: A twisted flip: $F(\Omega) \cong F(\Gamma)$.

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